Assignment 5 (Final Version)

Name		
Problem	**Points**	**Score**
1	6	
2	6	
3	10	
4	8	
5	6	
6	8	
7	8	
8	6	
9	10	
10	12	
11	6	
Total	**86**	

1. Section 1.3, Exercise 14.

Solution. Form the augmented matrix and reduce.

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= 2 - 5x_3 \\ x_2 &= 3 - 4x_3 \\ x_3 &= \text{free} \end{aligned}$$

To answer the question, The EF does not have a pivot in the rightmost column, so by Theorem 2 the system is consistent. In fact, there are infinitely many solutions: Using $x_3 = 0$, we get $2\mathbf{a}_1 + 3\mathbf{a}_2 = \mathbf{b}$.

2. Section 1.3, Exercise 21.

Solution. Form the augmented matrix and reduce.

$$\begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix} \sim \begin{bmatrix} \boxed{2} & 2 & h \\ 0 & \boxed{2} & \frac{h}{2} + k \end{bmatrix}$$

The EF does not have a pivot in the rightmost column, so by Theorem 2 the system is consistent and so $\begin{bmatrix} h \\ k \end{bmatrix} \in \text{Span}\{\mathbf{u}, \mathbf{v}\}$.

Another Method: By Theorem 4, there is a pivot in every **row** of the **coefficient matrix**, so the system is always consistent for any $\mathbf{b} = \begin{bmatrix} h \\ k \end{bmatrix}$.

3. Section 1.3, Exercise 26.

Solution. (a) Form the augmented matrix and reduce.

$$\begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & -2 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 3 & 5 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & \boxed{4} \end{bmatrix}$$

The EF has a pivot in the rightmost column, so by Theorem 2 the system is inconsistent and so $\mathbf{b} \notin W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

(b) The second column of A is in W, because it can be written as the following combination of the columns of A (no need to reduce!):

$$\begin{bmatrix} 0 \\ 8 \\ 2 \end{bmatrix} = 0 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 8 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}.$$

4. Section **1.4**, Exercise 12.

Solution. Form the augmented matrix corresponding to $A\mathbf{x} = \mathbf{b}$, and then reduce to RREF to find the solution.

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ -3 & -4 & 2 & 2 \\ 5 & 2 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & -8 & 8 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

So writing the solution as a vector, we get

$$\mathbf{x} = \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix} \text{ and } -4 \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}. \checkmark$$

5. Section 1.4, Exercise 14. Show your work.

Solution. By Theorem 3, $\mathbf{u}$ is in the subset spanned by the columns of A only if the corresponding matrix equation $A\mathbf{x} = \mathbf{u}$ has a solution. So row reduce $[A\,\mathbf{u}]$.

$$\begin{bmatrix} 2 & 5 & -1 & 4 \\ 0 & 1 & -1 & -1 \\ 1 & 2 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 & 0 & 4 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & 0 & \boxed{-3} \end{bmatrix} = EF.$$

By Theorem 2, since there is a pivot in the rightmost column, the system $A\mathbf{x} = \mathbf{u}$ is inconsistent. By Theorem 3, $\mathbf{u}$ is not in the span of the columns of A.

☞ NOTE: Theorem 4 does not apply. We are not asking "Is $A\mathbf{x} = \mathbf{b}$" consistent FOR ALL $\mathbf{b}$. We are asking whether $A\mathbf{x} = \mathbf{u}$ has a solution for a particular $\mathbf{u}$. Sometimes systems are consistent for SOME $\mathbf{b}$, but not other for others when there are NOT pivots in every row of the coefficient matrix A.

6. Section 1.4, Exercise 16. Ignore the instructions; instead, describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution. (Your description should be in the form of an equation involving b_1, b_2, and b_3). Also, give a specific example of a $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ does not have a solution, along with a few words of explanation.

Solution. By Theorem 2, the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution when we reduce $[A\,\mathbf{b}]$ and there is no pivot in the final column.

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ -2 & 2 & 0 & b_2 \\ 4 & -1 & 3 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & b_2 + 2b_1 \\ 0 & 7 & 7 & b_3 - 4b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & b_2 + 2b_1 \\ 0 & 7 & 7 & b_3 - 4b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & b_2 + 2b_1 \\ 0 & 0 & 0 & 3b_1 + 3.5b_2 + b_3 \end{bmatrix} = EF.$$

By Theorem 2, for the system to consistent there can be no pivot in the final column. So we need $3b_1 + 3.5b_2 + b_3 = 0$. A nicer way to write this is $6b_1 + 7b_2 + 2b_3 = 0$.

To create a specific $\mathbf{b}$ so that $A\mathbf{x} = \mathbf{b}$ has no solution, we just need to choose b_1, b_2, and b_3 so that $6b_1 + 7b_2 + 2b_3 \neq 0$. A simple example would be $b_1 = 0$, $b_2 = 0$, and $b_3 = 1$. Then $6b_1 + 7b_2 + 2b_3 = 2 \neq 0$. Substituting into the reduction above, we find

$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ -2 & 2 & 0 & 0 \\ 4 & -1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = EF.$$

The pivot in the final column means the system is not consistent by Theorem 2.

7. Find the value(s) of h for which $\mathbf{v} = \begin{bmatrix} -3 \\ h \\ -5 \\ 5 \end{bmatrix}$ is in Span $\left\{ \begin{bmatrix} -3 \\ -4 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix} \right\}$.

Show your work.

Solution. By Theorem 3, $\mathbf{v}$ is in the span of the given vectors only if the matrix equation $A\mathbf{x} = \mathbf{v}$ has a solution (no pivot in the final column). Form the augmented matrix and reduce to EF

$$\begin{bmatrix} -3 & 0 & 0 & -3 \\ -4 & 2 & 0 & h \\ 5 & -4 & 1 & -5 \\ -5 & 2 & -3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & h+4 \\ 0 & -4 & 1 & -10 \\ 0 & 2 & -3 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & h+4 \\ 0 & 0 & 1 & 2h-2 \\ 0 & 0 & -3 & 6-h \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{2} & 0 & h+4 \\ 0 & 0 & \boxed{1} & 2h-2 \\ 0 & 0 & 0 & \boxed{5h} \end{bmatrix} = EF.$$

By Theorem 2, for the system to consistent there can be no pivot in the final column. So we need $5h = 0$, or $h = 0$.

8. Section 1.4, Exercise 22. Show your work/explain your reasoning.

Solution. By Theorem 4, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$ if the matrix $A = [\mathbf{v}_1\, \mathbf{v}_2\, \mathbf{v}_3]$ has a pivot in every row. So reduce A to EF. (You do not need to go to all the way to RREF to determine where the pivots are located. But you must go to at least EF.)

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} \boxed{-3} & 9 & -6 \\ 0 & \boxed{-3} & -2 \\ 0 & 0 & \boxed{4} \end{bmatrix} = EF.$$

By Theorem 4, since there is a pivot in every row, Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$.

NOTE: It really does not make sense to try to use Theorem 2 here. We don't have an augmented matrix.

9. Section 1.4, Exercise 18. The instructions and the matrix B are located above Exercise 17. Justify your answer using an appropriate theorem.

Solution. (1) By Theorem 4, every vector in $\mathbb{R}^4$ is a linear combination of the columns of B if B has a pivot in every row. So reduce B to EF.

$$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 0 & 1 & 3 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 4 & 1 & 2 \\ 0 & \boxed{1} & 3 & -4 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} = EF.$$

There is no pivot in the final row, so by Theorem 4, every vector in $\mathbb{R}^4$ is NOT a linear combination of the columns of B.

(2) The columns of B do not span $\mathbb{R}^3$. The columns of B are vectors in $\mathbb{R}^4$ because they have four rows. vectors in $\mathbb{R}^3$ have only three rows. So the columns of B are not even in the set $\mathbb{R}^4$.

10. This question is designed to make you think about pivot positions in the rows and/or columns of a (coefficient) matrix A.

(*a*) Suppose A is a 4×4 matrix and $\mathbf{b} \in \mathbb{R}^4$ is a vector such that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Does the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ have a solution? If so, is the solution unique? Prove your answer very clearly, justifying your assertions very carefully.

Strategy: Use Theorem 2 to get a pivot in every COLUMN of A. Then use the size of A to get a pivot in every ROW. Then use Theorem 4.

Solution. Let A be a 4×4 matrix and $\mathbf{b} \in \mathbb{R}^4$ is a vector such that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Then by Theorem 2, the augmented matrix $[A\ b]$ has no pivot in the last column AND it has no free variables. So the *coefficient matrix* A has a pivot in every COLUMN. Since A has four columns, A has four pivots. Since A also has four rows, A has pivot in every ROW. Now by Theorem 4, the equation $A\mathbf{x} = \mathbf{c}$ has a solution for *all* $\mathbf{b} \in \mathbb{R}^4$, including $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. The solution will be unique because A has a pivot in every column.

(*b*) Suppose A is a 4×3 matrix and $\mathbf{b} \in \mathbb{R}^4$ is a vector such that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Does the equation $A\mathbf{x} = \mathbf{c}$ have a solution for *all* $\mathbf{c} \in \mathbb{R}^4$? Prove your answer very clearly, justifying your assertions very carefully. Use an appropriate theorem.

Solution. Quick proof: Let A be a 4×3 matrix. Since A has 4 rows, but only 3 columns, A has a maximum of 3 pivots. So A does not have a pivot in every row. By Theorem 4, $A\mathbf{x} = \mathbf{c}$ does not have a solution for all $\mathbf{c} \in \mathbb{R}^4$.

Solution. Another proof like part (a): Let A be a 4×3 matrix and $\mathbf{b} \in \mathbb{R}^4$ is a vector such that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Then by Theorem 2, the augmented matrix $[A\ b]$ has no pivot in the last column AND it has no free variables. So the *coefficient matrix* A has a pivot in every COLUMN. Since A has three columns, A has three pivots. Since A also has four rows, A DOES NOT HAVE a pivot in every ROW. Now by Theorem 4, the equation $A\mathbf{x} = \mathbf{c}$ does not always have a solution for *every* $\mathbf{c} \in \mathbb{R}^4$.

11. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 9 & 15 \\ 2 & 5 & h \end{bmatrix}$ For what values of h do the columns of A span $\mathbb{R}^3$? Be sure to show your work and justify your answer with an appropriate theorem.

Solution. (1) By Theorem 4, the columns of A span $\mathbb{R}^3$ if A has a pivot in every row. So reduce A to EF.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 9 & 15 \\ 2 & 5 & h \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 1 & h-6 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & \boxed{h-9} \end{bmatrix} = EF.$$

To ensure a pivot in every row, we need $h - 9 \neq 0$. By Theorem 4, for all values of h except 9, the columns of A span $\mathbb{R}^3$.