Assignment 6 (Part 1)

1. Practice writing proofs.

(*a*) Use the definition of scalar multiplication of vectors to prove: If $\mathbf{v} \in \mathbb{R}^n$, then $0\mathbf{v} = \mathbf{0}_n$, where the 0 on the left a scalar 0 and $\mathbf{0}_n$ on the right is the zero vector in $\mathbb{R}^n$.

Note: This is usually written as $0\mathbf{v} = \mathbf{0}$, without the subscript.

(*b*) Use part(a) and the DEFINITION of matrix-vector multiplication as a linear combination of the columns of A to prove: If A is an $m \times n$ matrix, then $A\mathbf{0}_n = \mathbf{0}_m$, where $\mathbf{0}_n$ on the left side of the equation is in $\mathbb{R}^n$ and $\mathbf{0}_m$ on the right side is in $\mathbb{R}^m$.

Note: This is usually written as $A\mathbf{0} = 0$ without the subscripts.

Proof. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$. Then $0\mathbf{v} = 0 \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0v_1 \\ \vdots \\ 0v_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$. □

Proof. Let $A = [\mathbf{a}_1 \ldots \mathbf{a}_n]$ be an $m \times n$ matrix. Then by definition of matrix-vector multiplication

$$A\mathbf{0}_n = [\mathbf{a}_1 \ldots \mathbf{a}_n] \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = 0\mathbf{a}_1 + \cdots + 0\mathbf{a}_n \overset{\text{Part (a)}}{=} \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}.$$

□

Name		
Problem	**Points**	**Score**
1	6	
2	7	
3	14	
4	24	
5	4	
Total	55	

2. Section 1.5, Exercise 8. Be careful. You are given the *coefficient* matrix, not the augmented matrix for homogeneous system $A\mathbf{x} = \mathbf{0}$.

Solution. Reduce the corresponding augmented matrix to RREF not just EF

$$\begin{bmatrix} 1 & -3 & -8 & 5 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -7 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= 2x_3 + 7x_4 \\ x_2 &= -2x_3 + 4x_4 \\ x_3 &= \text{free} \\ x_4 &= \text{free} \end{aligned}$$

So the general solution is

$$\mathbf{x} = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix} = s \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix} = s\mathbf{u} + t\mathbf{v}, \qquad s, t \in \mathbb{R}.$$

3. (*a*) Describe all solutions of the **homogeneous** system below on the left in parametric vector form. What type of geometric set does it form in $\mathbb{R}^4$?

$$\begin{aligned} x_1 + 3x_2 - 3x_3 + 7x_4 &= 0 \\ x_2 - 4x_3 + 5x_4 &= 0 \\ -2x_1 - 8x_2 + 14x_3 - 24x_4 &= 0 \end{aligned} \qquad \begin{aligned} x_1 + 3x_2 - 3x_3 + 7x_4 &= 1 \\ x_2 - 4x_3 + 5x_4 &= 2 \\ -2x_1 - 8x_2 + 14x_3 - 24x_4 &= -6 \end{aligned}$$

(*b*) Describe all solutions of the **non-homogeneous** system above on the right. Describe the geometry of the solution set compared to the solution set of part (a).

Solution. Reduce the corresponding augmented matrix to RREF (not just EF).

$$\begin{bmatrix} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \\ -2 & -8 & 14 & -24 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \\ 0 & -2 & 8 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & -8 & 0 \\ 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -9x_3 + 8x_4 \\ x_2 &= 4x_3 - 5x_4 \\ x_3 &= \text{free} \\ x_4 &= \text{free} \end{aligned}$$

So the general solution is

$$\mathbf{x} = x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix} = s \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix} = s\mathbf{u} + t\mathbf{v}, \qquad s, t \in \mathbb{R}.$$

Span $\{\mathbf{u}, \mathbf{v}\}$ is a plane in $\mathbb{R}^4$ through the origin (the origin corresponds to the trivial solution).

(*b*) For the nonhomogeneous system, reduce to RREF.

$$\begin{bmatrix} 1 & 3 & -3 & 7 & 1 \\ 0 & 1 & -4 & 5 & 2 \\ -2 & -8 & 14 & -24 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -3 & 7 & 1 \\ 0 & 1 & -4 & 5 & 2 \\ 0 & -2 & 8 & -10 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & -8 & -5 \\ 0 & 1 & -4 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -5 - 9x_3 + 8x_4 \\ x_2 &= 2 + 4x_3 - 5x_4 \\ x_3 &= \text{free} \\ x_4 &= \text{free} \end{aligned}$$

So the general solution is

$$\mathbf{x} = \begin{bmatrix} -5 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}, \qquad s, t \in \mathbb{R}.$$

The solution set is a plane in $\mathbb{R}^4$ that is translated parallel to the solution set of the homogenous system by the vector $\mathbf{p}$.

4. Answer each of the following. These questions make you think about Theorems 1.2 and 1.4 and about what we called Fact 2 in Friday's class (below). Give clear, careful, short **proofs** that your answers are correct, using theorems and facts.

FACT 2.1. The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

(*a*) Suppose A is a 5×5 matrix with 4 pivot positions.

1. Must the equation $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution?

Proof. Yes. Since there are only 4 pivots but 5 variables, there must be a free variable. By Fact 2 from Friday's class, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. □

2. Must the equation $A\mathbf{x} = \mathbf{b}$ have a solution for EVERY $\mathbf{b} \in \mathbb{R}^5$?

Proof. No. We are given that A has 4 pivots but it has 5 rows. So one row does not contain a pivot. By Theorem 4 (page 37) the system $A\mathbf{x} = \mathbf{b}$ does not have a solution for every $\mathbf{b} \in \mathbb{R}^5$ □

(b) Suppose A is a 5×4 matrix with 4 pivot positions.

1. Must the equation $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution?

Proof. No. Since there are 4 pivots and 4 variables, there is no free variable. By Fact 2 from Friday's class (or Theorem 2), the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. □

2. Must the equation $A\mathbf{x} = \mathbf{b}$ have a solution for EVERY $\mathbf{b} \in \mathbb{R}^5$?

Proof. No. We are given that A has 4 pivots but it has 5 rows. So one row does not contain a pivot. By Theorem 4 the system $A\mathbf{x} = \mathbf{b}$ does not have a solution for every $\mathbf{b} \in \mathbb{R}^5$. □

(c) Suppose A is a 4×5 matrix with 4 pivot positions.

1. Must the equation $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution?

Proof. Yes. Since there are only 4 pivots but 5 variables, there must be a free variable. By Fact 2 from Friday's class, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. □

2. Must the equation $A\mathbf{x} = \mathbf{b}$ have a solution for EVERY $\mathbf{b} \in \mathbb{R}^4$?

Proof. Yes. We are given that A has 4 pivots and it has 4 rows. Since every row has a pivot, by Theorem 4 the system $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^4$. □

5. Let $A = \begin{bmatrix} \mathbf{a}_1 \cdots \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 \cdots \mathbf{b}_n \end{bmatrix}$ both be $m \times n$ matrices. We define the **matrix sum** of A and B by adding the matrices column by column. That is,

$$A + B = \begin{bmatrix} (\mathbf{a}_1 + \mathbf{b}_1) \cdots (\mathbf{a}_n + \mathbf{b}_n) \end{bmatrix}.$$

Prove that matrix addition is commutative, that is, $A + B = B + A$, by using appropriate an vector property of $\mathbb{R}^m$.

Proof. Let $A = \begin{bmatrix} \mathbf{a}_1 \cdots \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 \cdots \mathbf{b}_n \end{bmatrix}$ both be $m \times n$ matrices. Then by definition of matrix addition,

Be careful: $\mathbf{a}_i$ and $\mathbf{b}_i$ are VECTORS, not real numbers!

$$\begin{aligned} A + B &= \begin{bmatrix} (\mathbf{a}_1 + \mathbf{b}_1) \cdots (\mathbf{a}_n + \mathbf{b}_n) \end{bmatrix} \\ &= \begin{bmatrix} (\mathbf{b}_1 + \mathbf{a}_1) \cdots (\mathbf{b}_n + \mathbf{a}_n) \end{bmatrix} \quad \text{Property (i) p. 27: vector addition commutes } \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \\ &= B + A \end{aligned}$$

□