Assignment 6 (Part 2)

Due Friday, February 12th in Class

☞ Remember WeBWorK set HW3 Due Monday, February 15th before the exam.

1. Section **1.7** Exercise 2.

Solution. To determine whether the vectors are independent, row reduce the augmented matrix of the homogeneous system $[\mathbf{v}_1\, \mathbf{v}_2\, \mathbf{v}_3\, \mathbf{0}]$.

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 2 & 0 & 3 & 0 \\ 3 & -8 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 3 & 0 \\ 3 & -8 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 3 & 0 \\ 0 & -8 & -3.5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The system has only the trivial solution $x_1 = x_2 = x_3 = 0$, so the vectors are independent by the Independence of Matrix Columns Theorem (p. 57) We could have stopped earlier at EF. There are pivots in every column of the *coefficient matrix* A, so there are no free variables, so the homogeneous system has only the trivial solution (Homogeneous Solutions Theorem (p. 43).

2. Section 1.7 Exercise 6.

Solution. To determine whether the column vectors are independent, row reduce the augmented matrix of the homogeneous system $[A\, 0]$.

$$\begin{bmatrix} -4 & -3 & 0 & 0 \\ 0 & -1 & 5 & 0 \\ 1 & 1 & -5 & 0 \\ 2 & 1 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} -4 & -3 & 0 & 0 \\ 0 & -1 & 5 & 0 \\ 1 & 1 & -5 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 1 & 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system has only the trivial solution $x_1 = x_2 = x_3 = x_4 = 0$, so the vectors are independent by the Independence of Matrix Columns Theorem (p. 57) We could have stopped earlier at EF. There are pivots in every column of the *coefficient matrix* A, so there are no free variables, so the homogeneous system has only the trivial solution (Homogeneous Solutions Theorem (p. 43).

Name		
Problem	**Points**	**Score**
1	6	
2	6	
3	7	
4	10	
5	12	
6	14	
Total	55	

3. Section 1.7 Exercise 12.

Solution. To determine when the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are dependent, row reduce the augmented matrix $[\mathbf{v}_1\, \mathbf{v}_2\, \mathbf{v}_3\, \mathbf{0}]$ and determine when there are non-trivial solutions.

$$\begin{bmatrix} 3 & -6 & 9 & 0 \\ -6 & 4 & h & 0 \\ 1 & -3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & 0 \\ 3 & -6 & 9 & 0 \\ -6 & 4 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -14 & h+18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & h+18 & 0 \end{bmatrix}$$

There will be non-trivial solutions only if there is no pivot in the third column, that is we need $h + 18 = 0$ or $h = -18$.

4. (*a*) Section 1.7, Exercises 16, 18, and 20. 'By inspection' means by simply looking at and thinking about the vectors, as opposed to using row-reduction. Justify your answer.

Solution. For Exercise 16, the columns are *dependent*. Notice $3\mathbf{v}_1 = -2\mathbf{v}_2$, so the vectors are scalar multiples of each other and are dependent. OR $3\mathbf{v}_1 + 2\mathbf{v}_2 = \mathbf{0}$.

For Exercise 18, the columns are *dependent*. There are more vectors than entries, so by the 'Surplus of Vectors Theorem' (Theorem 8), they are dependent.

For Exercise 20, the columns are *dependent*. The set contains the zero vector, $\mathbf{0}$. By Theorem 9 they are dependent.

(*b*) Now think about the two vectors in Exercise 16 as forming a matrix

$$A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \\ 8 & -12 \end{bmatrix}.$$

Give one non-trivial solution to the system $A\mathbf{x} = \mathbf{0}$ by inspection.

Solution. In part (a) we noted that $3\mathbf{v}_1 + 2\mathbf{v}_2 = \mathbf{0}$. So one obvious solution is $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Because then:

$$A\mathbf{x} = \begin{bmatrix} 2 & -3 \\ -4 & 6 \\ 8 & -12 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \, \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3\mathbf{v}_1 + 2\mathbf{v}_2 = \mathbf{0}.$$

5. (*a*) True or False: The columns of any 4×5 matrix A are linearly *dependent*. Explain.

Solution. True. Use the 'Surplus of Vectors Theorem' (Theorem 8). There are more vectors than entries ($p = 5$, $n = 4$).

(*b*) How many pivot columns must a 6×4 matrix A have if its columns are linearly independent? Justify/explain your answer carefully.

Solution. Four. The the system $A\mathbf{x} = 0$ (which is always consistent) must have only the non-trivial solution, so there can be no free variables. (Independence of Matrix Columns (p. 57) iff $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.)

(*c*) How many pivot columns must a 4×6 matrix A have if its columns span $\mathbb{R}^4$? Justify/explain your answer carefully. What theorem applies?

Solution. Theorem 4 applies. By this theorem the columns of A span $\mathbb{R}^4$ if and only if there is a pivot in every row. So there must be 4 pivots.

(*d*) Suppose that A is an $m \times n$ matrix with the property that $A\mathbf{x} = \mathbf{b}$ has at most one solution. Prove that the columns of A are linearly independent.

☞ This question was poorly phrased.

I meant to say: Suppose that A is an $m \times n$ matrix with the property that *for all* $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has at most one solution. Prove that the columns of A are linearly independent. This the question I have answered. I took any reasonable interpretation.

Solution. Let $b = \mathbf{0}$. Since $A\mathbf{x} = \mathbf{b} = \mathbf{0}$ has at most one solution and it is always consistent, the matrix A must have pivots in every column (no free variables). So the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This means the columns are independent. (Independence of Matrix Columns (p. 57) iff $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.)

6. Basic computations: Let $A = \begin{bmatrix} 1 & 2 & 5 & -1 \\ 3 & 1 & 5 & 2 \\ 2 & 0 & 2 & 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}$.

(*a*) Determine $3\mathbf{v}$.

Solution. $3\mathbf{v} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \\ -3 \end{bmatrix}.$

(*b*) Use the definition of a matrix-vector product on page 35 to compute $A\mathbf{v}$.

Solution.

$$A\mathbf{v} = \begin{bmatrix} 1 & 2 & 5 & -1 \\ 3 & 1 & 5 & 2 \\ 2 & 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \\ 2 \end{bmatrix}.$$

(*c*) Find the general solution to $A\mathbf{x} = 0$.

Solution. Reduce $[A\,\mathbf{0}]$.

$$\begin{bmatrix}1&2&5&-1&0\\3&1&5&2&0\\2&0&2&2&0\end{bmatrix} \sim \begin{bmatrix}1&0&1&1&0\\1&2&5&-1&0\\3&1&5&2&0\end{bmatrix} \sim \begin{bmatrix}1&0&1&1&0\\0&2&4&-2&0\\0&1&2&-1&0\end{bmatrix} \sim \begin{bmatrix}1&0&1&1&0\\0&1&2&-1&0\\0&0&0&0&0\end{bmatrix} \quad \begin{aligned}x_1 &= -x_3 - x_4\\ x_2 &= -2x_3 + x_4\\ x_3 &= \text{free}\\ x_4 &= \text{free}\end{aligned}$$

So parametrically,

$$\mathbf{x} = x_1 \begin{bmatrix}-1\\-2\\1\\0\end{bmatrix} + x_2 \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}.$$

(*d*) Use the previous part to find two non-trivial solutions $\mathbf{w}_1$ and $\mathbf{w}_2$ to the system. For your choice of $\mathbf{w}_1$ verify $A\mathbf{w}_1 = \mathbf{0}$.

Solution. Let $x_3 = 1$ and $x_4 = 0$ to get $\mathbf{w}_1 = \begin{bmatrix}-1\\-2\\1\\0\end{bmatrix}$. Let $x_3 = 0$ and $x_4 = 1$ to get $\mathbf{w}_2 = \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}$. Then

$$A\mathbf{w}_2 = \begin{bmatrix}1&2&5&-1\\3&1&5&2\\2&0&2&2\end{bmatrix}\begin{bmatrix}-1\\1\\0\\1\end{bmatrix} = -1\begin{bmatrix}1\\3\\2\end{bmatrix} + 1\begin{bmatrix}2\\1\\0\end{bmatrix} + 0\begin{bmatrix}5\\5\\2\end{bmatrix} + 1\begin{bmatrix}-1\\2\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

(*e*) Do the columns of A span $\mathbb{R}^3$? Explain. (You may be able to re-use earlier work in this problem.)

Solution. When we reduced A to RREF, there were only two pivots. But there are three rows. Not every row has pivot, so by Theorem 4 (Spanning and Pivots Theorem) the columns do not span $\mathbb{R}^3$.

(*f*) Are the columns of A independent? (You may be able to re-use earlier work in this problem, but there is another easy method.)

Solution. Method 1: There are more vectors (4) than entries (3). So by the Surplus of Vectors Theorem, the columns are *dependent*.

Method 2: We saw in part (c) that there were non-trivial solutions (free variables) to the homogeneous system $A\mathbf{x} = \mathbf{0}$. This means that the columns of A are dependent. (Independence of Matrix Columns (p. 57) iff $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.)