PROOF GRAMMAR: Suppose we to prove the following made up theorem.

THEOREM. If A is a green watchamajig, then A is an upside-down what-ever.

A good proof (better than those in your text) would have the following form (read note in margin!):

Proof. Assume A green watchamajigr. (Show A is an upside-down what-ever.) Since A watchamajig, by definition it has special property X. By Theorem 28, If A has property X, then it also has property Y. But if A has property Y, then it is a what-ever. Because A is green, by Theorem 84 it is also upside-down. Therefore, we see that A is an upside-down what-ever. □

☞. The proof starts with the word 'proof' and should restate the hypothesis. (It is also useful to state parenthetically what you are trying to prove.) Be sure to mention the theorems/definitions/facts as you use them. Finish with a little box to indicate that the proof is over.

1. Prove each of the following theorems using the format and structure of the "proof" above.

THEOREM 0.0.1. Assume that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with standard $m \times n$ matrix A. If T is onto, then $m \leq n$.

THEOREM 0.0.2. Assume that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with standard $m \times n$ matrix A. If T is one-to-one, then $m \geq n$.

THEOREM 0.0.3. Assume that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with standard $m \times n$ matrix A. If T is both onto and one-to-one, then $m = n$.

2. Recall that the contrapositive of the statement: "If P, then Q" is the statement "$If \sim Q$, then $\sim P$." State the contrapositive of Theorems 1 and 2 above.

3. Consider the matrices A defined in Exercises 4, 6, and 10 of Section 1.8 and do the following for each one. (Ignore the text instructions, and ignore the given vector $\mathbf{b}$.)

(*a*) Decide whether the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ in question is onto its co-domain. Justify your answers with an appropriate theorem.

(*b*) Decide whether the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ in question is one-to-one. Justify your answers with an appropriate theorem.

Note: If any of these problems require row-reducing a matrix, you may use Maple or some other such program to do the calculation. You should give the original matrix and the reduced matrix.

4. Consider the transformations defined in Exercises 17, 19, and 20 of Section 1.9. For each one, show that it is not one-to-one by giving two input vectors that have the same image. You should be able to find the vectors "by inspection." However your answer should include a calculation that your verifying that your vectors "work."

5. Decide whether or not each of the following functions is one-to-one. Justify your answers. In particular, if you claim that a function is not one-to-one, you must justify that there are two different inputs that go to the same output.

(*a*) The function f that takes as input a living person and outputs the current age in years of that person.

(*b*) The function g that takes as input a country and outputs the capital city of that country.

(*c*) The function h that takes a month as input and outputs are the number of days in the month

6. (*a*) Consider the function in part (a) of the preceding question. Is the function onto the set of natural numbers (the set of all positive integers)? Justify your answer carefully.

(*b*) Consider the function in part (b) of the preceding question. Is the function onto the set of all the world's cities? Justify your answer carefully.

(*c*) Consider the function in part (c) of the preceding question. Is the function onto the set of all natural numbers? Justify your answer carefully.

7. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Can there be a vector $\mathbf{b}$ in $\mathbb{R}^m$ such that T sends *exactly* two vectors to $\mathbf{b}$? Justify your answer with appropriate theorem(s).

8. Section 1.8, Exercise 30. Hint: If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$, how can we rewrite the vector $\mathbf{x}$?

9. Real world applications: Read Example 1 (and its pre-amble) of Section 1.10. Then do Exercises 2 and 4 on page 86. Use `Maple`.