Assignment 9. One of the goals of this course is to have you become more adept at 'doing proofs.' The first three questions below just require using the definitions of the terms involved. When you want to to prove that something is an XYZ, then you need to verify that it has the defining property of an XYZ.

1. One of the concepts we will encounter later in the term is the kernel of a linear transformation. **Definition:** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear. The **kernel** of T is the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{0}$

(*a*) Prove that the kernel of T is closed under addition: that is, if $\mathbf{x}$ and $\mathbf{y}$ are both in the kernel of T, then so is the vector $\mathbf{x} + \mathbf{y}$.

(*b*) Prove that the kernel of T is closed under scalar multiplication: that is, if $\mathbf{x}$ is in the kernel of T, then so is the vector $c\mathbf{x}$ for any scalar c.

2. This problem asks you to think about the definition of range. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are both in the range of T.

(*a*) Prove that the vector $\mathbf{a} + \mathbf{b}$ is also in the range of T. (This shows that the range of T is closed under addition.)

(*b*) Is the range of T closed under scalar multiplication: that is, if $\mathbf{a}$ is in the range of T, then so is the vector $c\mathbf{a}$ for any scalar c.

3. This is not hard; you just have to be careful. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ both be linear transformations. (Note the sizes!) Define the composite transformation by $G : \mathbb{R}^n \to \mathbb{R}^p$ by $G(\mathbf{x}) = S(T(\mathbf{x}))$. Prove that G is a linear transformation. (Check the two properties using the definition of G.)

4. Basic Concept Check: Consider the sets $X = \{1, 2, 3\}$ and $Y = \{a, b\}$. Give all functions from X to Y. Describe a function by specifying $f(1) = \ldots$, $f(2) = \ldots$, and $f(3) = \ldots$. List them in four separate groups: Note: there should be 8 different functions in total.

(*a*) those that are both onto Y and one-to-one;

(*b*) those that are onto Y but not one-to-one;

(*c*) those that are one-to-one but are not onto Y;

(*d*) those that are neither onto Y and not one-to-one.

5. OK, I need to make sure you can do some basic matrix computations. Section 2.1 Exercises 2, 6 (two ways), 10 and 12. Exercise 10 shows that we can't "cancel" when multiplying by matrices: If $AB = AC$ we cannot conclude that $B = C$. For Exercise 12, think about the relation between the columns of A. This problem uses the same ideas as Problem 7.

6. Using definitions.

(*a*) Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix, and that B's first two columns are equal. Use the definition of matrix multiplication (page 95) to prove that the first two columns of AB are equal.

(*b*) Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix, and that B's third column is a linear combination of its first two columns. Use the definition of matrix multiplication, and properties of matrix multiplication, to prove that the third column of AB is a linear combination of the first two columns of AB. (Use the definition of linear combination to rewrite $\mathbf{b}_3$.)

7. Order of operations. Prove the following result using an appropriate theorem: If A be an $n \times n$ matrix, then $(A^2)^T = (A^T)^2$.

Day 15: Reading and Practice

Office Hour Help: M & W 2:30–4:00, Tu 2:00–3:30, & F 1:30–2:30 or by appointment. Website: `http://math.hws.edu/~mitchell/Math204S16/index.php`.

1. Read Section 2.2 and Review Section 2.1 which we should finish next time. This is basic matrix algebra and I think we can cover much of it quickly.

(*a*) Key terms and concepts: $n \times n$ identity matrix, $m \times n$ zero matrix, diagonal entries, $n \times n$ diagonal matrix, matrix multiplication (columns of AB are linear combinations of the columns of A), *matrix multiplication is not commutative*, A^T the transpose of A. Next up: multiplicative inverses of square matrices *sometimes* exist.

2. Practice. Not to be handed in, check the answers in the back.

(*a*) Page 100: #5, 7, 8, 9, 10, 31.

(*b*) Try #17. See the discussion after Example 3. First you are trying to solve $A\mathbf{x} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$.

(*c*) Try #23. Given $A\mathbf{x} = \mathbf{0}$. Multiply both sides of this equation on the LEFT by C. Now use the other piece of information in the problem.

(*d*) Suppose that A is $m \times n$ and that $CA = I_n$. What size is C? Now suppose that $AD = I_m$. What size is D? Compute CAD two ways: as $(CA)D$ and $C(AD)$. So what can you say about about the matrices C and D? What can you say about their sizes? Now what can you say about the size of A? Now do problem #25.

Hand In Wednesday

☞ Begin WeBWorK LHW6 online, due Thursday. Work on the problem set due Friday (see back of page). Come to the talk on Wednesday.

1. Assume B is a 3×3 matrix and that $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 12 \\ -1 & -2 & -3 \end{bmatrix}$. Suppose that $AB = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix}$. Determine the three columns of B as follows.

(*a*) Solve the system of equations $A\mathbf{x} = \mathbf{e}_1$, where, as usual, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(*b*) Now solve the systems $A\mathbf{y} = \mathbf{e}_2$ and $A\mathbf{z} = \mathbf{e}_3$.

(*c*) Using the vectors you found above, form the matrix $B = [\mathbf{x}\ \mathbf{y}\ \mathbf{z}]$. Then calculate AB and BA.

(*d*) What's interesting about your answers above? How would you describe the relationship between A and B?

(*e*) Use a theorem in Section 2.1 to quickly evaluate $A^T B^T$ and $B^T A^T$ without and further numerical calculations.