Assignment 13

Due Wednesday after break. Prove the following theorems.

1. Prove: If A is $n \times n$, then $\det A^T = \det A$. (This means we can do column operations on a matrix to simplify calculating the determinant.)

2. Prove: If A is $n \times n$ and invertible, then $\det A^{-1} = \dfrac{1}{\det A} = (\det A)^{-1}$.

3. Prove: If A is square matrix n is a positive integer, then $\det A^n = (\det A)^n$. If you know how to do induction, try it that way.

4. Let A is $n \times n$ and let k be any scalar. Find a formula for $\det(kA)$ in terms of $\det A$.

5. A $n \times n$ invertible matrix U is **orthogonal** if $U^{-1} = U^T$. Prove that if U is orthogonal, then $\det U = \pm 1$.

6. An $n \times n$ matrix A is said to be **conjugate** to the $n \times n$ matrix B if there is some invertible $n \times n$ matrix M so that $A = MBM^{-1}$. Prove: If A is conjugate to B, then $\det A = \det B$.

7. Page 176, #40.

8. **Sebastien's Theorem.** Consider these three **off-diagonal** matrices.

$$A = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & d & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 \\ 0 & f & 0 & 0 & 0 & 0 \\ g & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In each case, is the determinant of each the product of the diagonal elements? Show your work, justify your answer. Hint: Use row operations.
☞ Bonus: Can you figure out for which size matrices Sebastien's Theorem holds?

———————— Added Monday ————————

9. Section 3.3 (Page 184–5) Exercises #20, 22, and 28.

10. Suppose that $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$. If $\det A = -3$, evaluate $\det \begin{bmatrix} a & b & c-2a & d \\ e & f & g-2e & h \\ i & j & k-2i & l \\ m & n & o-2m & p \end{bmatrix}$.

11. Bonus: Find a formula (using determinants) for the area of a triangle whose vertices are $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$. Explain why your formula is correct.