Systems of Linear Equations

DEFINITION 1.1.1. A linear equation in the variables $x_1, x_2, \ldots, x_n$ can be written in the form

$$a_1x_1 + a_2x_2 + a_nx_n = b$$

where b and the coefficients $a_1, \ldots, a_n$ are real (or complex) numbers.

EXAMPLE 1.1.2. A simple example: $9x_1 + 1.5x_2 - \pi x_3 = 9$. But also consider the equations

$$4x_1 - 5x_2 + 2 = x_1 \qquad \text{and} \qquad x_2 = 2(\sqrt{6} - x_1) + x_3$$

each of which can be re-arranged to linear equations:

$$3x_1 - 5x_2 = -2 \qquad \text{and} \qquad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

Compare with these which are not linear:

$$4x_1 - 6x_2 - x_1x_2 = 0 \qquad \text{and} \qquad x_1 + 2\sqrt{x_2} = -7.$$

DEFINITION 1.1.3. A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same set of variables, say, $x_1, x_2, \ldots, x_n$.

For example,

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 10 \\ x_1 \qquad - 4x_3 &= -2 \end{aligned}$$

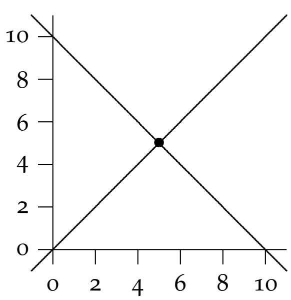
DEFINITION 1.1.4. A solution of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

For example, $(2, 3, 1)$ and $(-2, 14, 0)$ are both solutions to the system above (check).

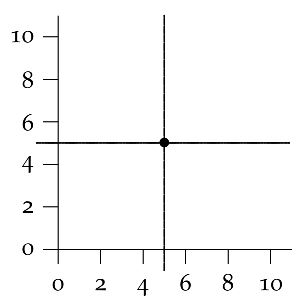
DEFINITION 1.1.5. The set of all possible solutions to a linear system is called the **solution set**. Two linear systems are **equivalent** if they have the same solution set.

EXAMPLE 1.1.6 (Linear Systems Two Equations in Two Variables)**.** Solving such a linear system amounts to finding the intersection of two lines. There are three possibilities.

$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned}$$

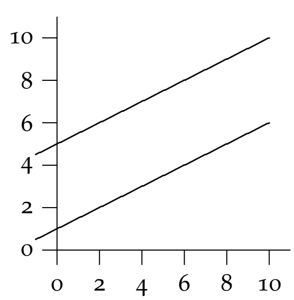


$$\begin{aligned} x_1 \quad &= 5 \\ x_2 &= 5 \end{aligned}$$



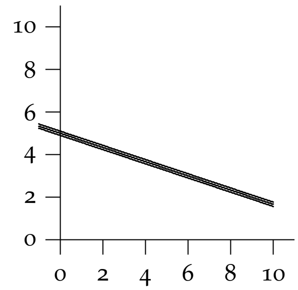
The top two systems are equivalent since they have the same solution set, namely $\{(5, 5)\}$.

$$\begin{aligned} x_1 - 2x_2 &= -10 \\ x_1 - 2x_2 &= -2 \end{aligned}$$



This system has no solution because the lines are parallel.

$$\begin{aligned} x_1 + 3x_2 &= 15 \\ -2x_1 - 6x_2 &= -30 \end{aligned}$$



This system has infinitely many solutions because the lines are the same.

This illustrates a general fact that we will prove in a few days.

THEOREM 1.1.7. A system of linear equations exactly one of the following is true:

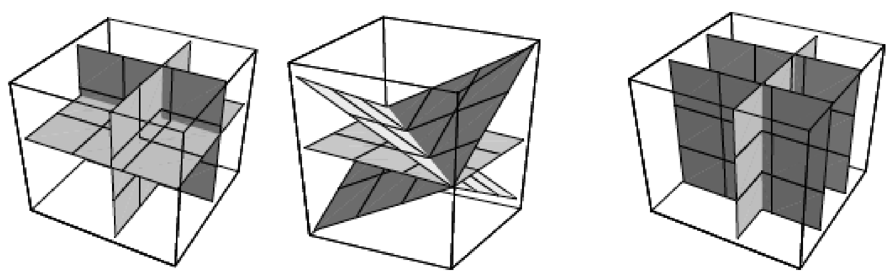
(1) it has no solution (inconsistent);
(2) it has exactly one solution (consistent);
(3) it has infinitely many solutions (consistent).

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

i) The planes intersect in one point. *(one solution)*

ii) The planes intersect in one line. *(infinitely many solutions)*

iii) There is not point in common to all three planes. *(no solution)*



Matrix Notation

We can capture the essential information about a linear system in a rectangular array of numbers called a **matrix**. If we start with the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ -x_1 + 4x_2 - 9x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned} \tag{1.1}$$

we can form the **coefficient matrix** and the **augmented matrix** for the system:

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 4 & -9 \\ 5 & 0 & -5 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ -1 & 4 & -9 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

The **size** of the matrix tells us how many rows and columns it has. The augmented matrix for this system has 3 rows and 4 columns so is a 3×4 matrix. More generally, an $\mathbf{m} \times \mathbf{n}$ matrix has m rows and n columns (where m and n are positive integers).

Solving a Linear System Efficiently

We will be solving lots of linear systems this semester. We need to be able to do it efficiently using a process that we know will always work. The key idea is to *replace one system with an **equivalent** system that is easier to solve.* There are three basic operations used to do this:

(1) Replace an equation by adding to it a multiple of another equation.
(2) Interchange the order of two equations.
(3) Replace an equation by a non-zero multiple of itself.

Remember, equivalent systems have the same solution sets, as in the first part of Example 1.1.6.

NOTE: These operations do not change the solution set of the system because they involve doing the same thing to each side of one or more of the equations, as the example below illustrates.

EXAMPLE 1.1.8. Solve the system in (1.1). We solve the system using an elimination process.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ -x_1 + 4x_2 - 9x_3 &= 8 \\ 5x_1 \quad - 5x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ -1 & 4 & -9 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \tag{1.2}$$

We solve the system side-by-side with the augmented matrix. In the future, we would use only the augmented matrix, which simplifies the process.

Keep x_1 in the first equation and eliminate it from the others. To Row 2 add Row 1:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \quad - 5x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ & & & \\ & & & \end{bmatrix} \tag{1.3}$$

Indicate this by using $R_2 + R_1 \to R_2$.

To Row 3 add -5 times Row 1.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 10x_2 \quad &= \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & & \end{bmatrix} \tag{1.4}$$

$R_3 - 5R_1 \to R_3$.

Multiply Row 2 add $\frac{1}{2}$ to obtain a 1 as the coefficient of x_2.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 \quad & \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ & & & \\ 0 & 10 & -10 & 10 \end{bmatrix} \tag{1.5}$$

$\frac{1}{2}R_2 \to R_2$.

Use the x_2 in equation 2 to eliminate x_2 from equation 3. To Row 3 subtract 10 times Row 2.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & & \end{bmatrix} \tag{1.6}$$

$R_3 - 10R_2 \to R_3$.

Multiply Row 3 add $\frac{1}{30}$ to obtain a 1 as the coefficient of x_3.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{1.7}$$

$\frac{1}{30}R_3 \to R_3$.

The matrix is now in **triangular form**. Now life is easy. At this stage we know the system is *consistent* and can solve it. We know $x_3 = -1$. We can put this into the second equation and solve for x_2 and then use the two values to solve for x_1. However, we will take a slightly different approach.

Eliminate the x_3 terms from equations 1 and 2 by using multiples of equation 3. To Row 2 add 4 times Row 3, then from Row 1 subtract Row 3.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 \quad &= \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & & \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad \begin{aligned} x_1 - 2x_2 \quad &= \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & & \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{1.8}$$

$R_2 + 4R_3 \to R_2$.
$R_1 - R_3 \to R_1$.

Now eliminate x_2 in Row 1: To Row 1 add 2 times Row 2.

$$\begin{aligned} x_1 \quad &= 1 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{1.9}$$

$R_1 + 2R_2 \to R_1$.

So the *only* solution to the original system is $(1, 0, -1)$. Let's check this:

$$\begin{array}{llll} 1(1) & -2(0) & +1(-1) & = 0 \checkmark \\ -1(1) & +4(0) & -9(-1) & = 8 \checkmark \\ 5(1) & & -5(-1) & = 10 \checkmark \end{array} \tag{1.10}$$

In the future we will use only the augmented matrix to solve the system, because it is easy to move back and forth between an augmented matrix and the corresponding system. The key to solving the system was using

DEFINITION 1.1.9. The three **elementary row operations** are

1. (Replacement) Add a multiple of one row (equation) to another.
2. (Scale) Multiply a row (equation) by a nonzero constant.
3. (Interchange) Interchange the order of two rows (equations).

In the first two cases we are essentially 'performing the same mathematical operation to both sides of a given equation,' so the solution does not change.

NOTE: These operations are *reversible*. Repeating an interchange undoes the original interchange, scaling a row by c is undone by scaling by $\frac{1}{c}$, and adding a multiple c of one row to another is undone by adding a multiple $-c$ times the row to the replaced row.

DEFINITION 1.1.10. Two matrices are **row equivalent** if there is a sequence of row operations that transforms one into the other.

For those of you who have taken Math 135 or CS 229, row equivalence is an equivalence relation. It is reflexive, symmetric, and transitive.

Because row operations are reversible, if matrix A is row equivalent to B, then B is row equivalent to A.

Most important, any solution of the original system remains a solution of the row equivalent system. And since the new system can be converted back to the original one by row operations, any solution of the new system is a solution of the old system. Thus,

THEOREM 1.1.11. Row equivalent linear systems have the same solution set.

TWO FUNDAMENTAL QUESTIONS ABOUT LINEAR SYSTEMS.

1. Is the system is consistent; that is, does at least one solution *exist*?
2. If a solution exists, is it the only one; that is, is the solution *unique*?

EXAMPLE 1.1.12. Determine if the following system is consistent.

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned} \qquad \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2 \quad \begin{bmatrix} & & & \\ & & & \\ 4 & -8 & 12 & 1 \end{bmatrix} \rightarrow$$

$$R_3 - 2R_1 \rightarrow R_3 \quad \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix} \quad R_3 + 2R_2 \rightarrow R_3 \quad \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ & & & \end{bmatrix}$$

which is equivalent to the linear system

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= \end{aligned}$$

The last equation shows that the system is inconsistent (no solutions). This type of last row is typical in an inconsistent system.

EXERCISE 1.1.13. For what values of h will the following system be consistent? $\begin{aligned} 3x_1 - 9x_2 &= 4 \\ -2x_1 + 6x_2 &= h \end{aligned}$

Solution. Strategy: Reduce to triangular form.

$$\begin{bmatrix} 3 & -9 & 4 \\ -2 & 6 & h \end{bmatrix} \tfrac{1}{3}R_1 \rightarrow R_1 \begin{bmatrix} & & \\ -2 & 6 & h \end{bmatrix} R_2 + 2R_1 \rightarrow R_2 \begin{bmatrix} 1 & -3 & \frac{4}{3} \\ & & \end{bmatrix}$$

What must h be for the system to be consistent?

Classwork: Modeling A Closed Economy

Suppose a student wants to create a simple model of a closed economy, that is, whatever is produced by the various sectors is also entirely consumed by these sectors of the economy—there is no external consumption. The sectors are Coal, Electric, and Steel. The output of each sector is consumed by these sectors according to this table.

Consumed By	Output Produced By		
	Coal	Electric	Steel
Coal	0.0	0.5	0.5
Electric	0.4	0.4	0.3
Steel	0.6	0.1	0.2

1. (*a*) Interpret the meaning of third row of the table.

(*b*) Interpret the meaning of second column of the table.

2. Why do the columns sum to 1? Why don't the rows?

3. Denote the prices (i.e., say in billions of dollars) of the Coal, Electric, and Steel sectors by x_1, x_2, and x_3, respectively. If possible, find *equilibrium prices* that make each sector's income match its expenditures. (This is type of pricing is what must happen in a communal setting where all goods and resources are shared equally. Something like this was attempted in the former Soviet Union.)

(*a*) How can you use the rows of the table to find relations among the prices?

(*b*) What system of equations do you obtain?

(*c*) Can you solve this system?

Solution. Since each sector's expenses must match its income, the corresponding linear system is

$$\begin{aligned} \text{Expense} \quad &= \text{Income} \\ 0.5x_2 + 0.5x_3 &= x_1 \\ 0.4x_1 + 0.4x_2 + 0.3x_3 &= x_2 \\ 0.6x_1 + 0.1x_2 + 0.2x_3 &= x_3 \end{aligned}$$

We convert this to standard form and then create and reduce the corresponding augmented matrix. *Fill in the entries rows.*

$$\begin{aligned} -x_1 + 0.5x_2 + 0.5x_3 &= 0 \\ 0.4x_1 - 0.6x_2 + 0.3x_3 &= 0 \\ 0.6x_1 + 0.1x_2 - 0.8x_3 &= 0 \end{aligned} \quad \begin{bmatrix} -1 & 0.5 & 0.5 & 0 \\ 0.4 & -0.6 & 0.3 & 0 \\ 0.6 & 0.1 & -0.8 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0.5 & 0.5 & 0 \\ 0 & -0.4 & 0.5 & 0 \\ & & & \end{bmatrix} \to$$

$R_2 + 0.4R_1 \to R_2$ and $R_3 + 0.6R_1 \to R_3$

$$\begin{bmatrix} -1 & 0.5 & 0.5 & 0 \\ 0 & -0.4 & 0.5 & 0 \\ & & & \end{bmatrix} \to \begin{bmatrix} -1 & 0.5 & 0.5 & 0 \\ 0 & 1 & -5/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & & 0 \\ 0 & 1 & -5/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & -5/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 + R_2 \to R_3$,
then $-\frac{10}{4}R_2 \to R_2$,
then $R_1 - 0.5R_2 \to R_1$,
then $-R_1 \to R_1$

Interpret the answer: