Row Reduction and Echelon Forms

DEFINITION 1.1.1. A matrix is in **echelon form** or **row echelon form** if

(1) All nonzero rows are above any rows of all zeros.

(2) Each leading entry (i.e., left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.

(3) All entries in a column below a leading entry are zero.

The matrix is in **reduced echelon form** if it satisfies the additional conditions

(4) The leading entry in each nonzero row is 1.

(5) Each leading 1 is the only nonzero entry in its column.

(*a*) Determine which of these **augmented** matrices are in echelon form (but not reduced row echelon form), which are in RREF, and which are neither.

(*b*) Since these are augmented matrices, which systems in EF or RREF are not consistent?

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 6 & 3 \\ 0 & 1 & -6 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 2 & 0 & 0 & -3 & 0 & 7 \\ 1 & 0 & 0 & 0 & 1 & 0 & 6 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 9 & 0 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 8 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 0 & 2 & 6 \\ 0 & 1 & 9 & 8 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 6 & 0 & 0 & 4 & 0 & 11 \\ 0 & 8 & 0 & 1 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 12 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 4 & 0 & 11 \\ 0 & 1 & 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 12 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 6 \end{bmatrix}$$

KEY POINT: Reduced row echelon form is that it allows us to read off the answer to the system easily. Note: An augmented matrix A has lots of echelon forms BUT

FACT 1.1. Each matrix is row equivalent to exactly one matrix in reduced row echelon form. (Every matrix has a unique reduced echelon form.)

Pivots and Echelon Form

Pivot positions and columns in a matrix correspond to the positions and columns of the leading 1's in the reduced echelon form of a matrix.

DEFINITION 1.1.2. A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced row echelon form of A. A **pivot column** is a column that contains a pivot position.

NOTE: There is at most one pivot in any row or in any column. Why?

EXERCISE: Identify the pivot positions and columns in the examples above. Notice that they can also be determined from the echelon form of the matrix.

EXAMPLE 1.1.3. Reduce to echelon form and locate the pivot columns of A below.

$$A = \begin{bmatrix} 0 & 5 & 10 & -15 & -15 \\ 0 & 2 & 4 & -6 & -6 \\ 1 & 4 & 5 & -9 & -7 \\ 2 & 2 & -2 & -10 & 4 \end{bmatrix}$$

SOLUTION. To do the reduction, we want to move a non-zero entry into the upper-left position. Such a non-zero entry in a pivot position is called a **pivot**. Move the third row to the top because it starts with a 1, which we can then use to easily clear out the entries below it. So interchange the first and third rows. The first pivot is boxed.

$$R_3 \longleftrightarrow R_1 \qquad \begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 2 & 2 & -2 & -10 & 4 \end{bmatrix}$$

Create 0's below the pivot 1 by adding multiples of the first row to the later rows. The pivot position in the second row must be as far left as possible—here, this will be the second column. The 2 is a reasonable choice for the pivot, since there are no 1's available in the second column in later rows. So 2 is the pivot for the second column.

$$\begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 2 & 2 & -2 & -10 & 4 \end{bmatrix} \quad R_4 - 2R_1 \to R_4 \quad \begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -6 & -12 & 8 & 18 \end{bmatrix}$$

Now clear the entries below this pivot: $R_3 \to R_3 - \frac{5}{2}R_2$ and $R_4 \to R_4 + 3R_33$.

$$\begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & & & & \\ 0 & & & & \end{bmatrix}$$

We have a row of 0's so we cannot create a pivot in R_3. (Using rows 1 or 2 would destroy the echelon form.) But we can interchange R_3 and R_4.

$$\begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & \boxed{-10} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in echelon form. The pivot columns are: ____________________
The pivots are: ____

The Row Reduction Algorithm

The algorithm consists of 4 steps to get to EF (echelon form), and 5 steps to RREF.

THE ROW REDUCTION ALGORITHM.

1. Step 1. Begin with leftmost nonzero column (1st pivot column)
2. Step 2. Select a nonzero entry in the pivot column and exchange rows if needed to move it to the pivot position.
3. Step 3. Use row operations to get 0's below the pivot position.
4. Step 4. Ignore the pivot row (and rows above it). Repeat Steps 1–3 until EF is reached.
5. Step 5. Back-substitute (described below) to get to RREF.

EXAMPLE 1.1.4. Apply elementary row operations to transform the matrix below to RREF.

$$\begin{bmatrix} 0 & 4 & 4 & 8 & 2 & 4 \\ 3 & 1 & 7 & 11 & 0 & 6 \\ 2 & 2 & 6 & 10 & 1 & 8 \end{bmatrix}$$

Steps 1 and 2: Get a nonzero entry in the 1st pivot position. [We also scale the new top row to have a leading 1.]

$$\begin{bmatrix} 0 & 4 & 4 & 8 & 2 & 4 \\ 3 & 1 & 7 & 11 & 0 & 6 \\ 2 & 2 & 6 & 10 & 1 & 8 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 6 & 10 & 1 & 8 \\ 3 & 1 & 7 & 11 & 0 & 6 \\ 0 & 4 & 4 & 8 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & & & & & \\ 3 & 1 & 7 & 11 & 0 & 6 \\ 0 & 4 & 4 & 8 & 2 & 4 \end{bmatrix}$$

$R_3 \longleftrightarrow R_1$

$\frac{1}{2}R_1 \to R_1$

Step 3: Clear out below the pivot.

$$\sim \begin{bmatrix} 1 & 1 & 3 & 5 & 1/2 & 4 \\ 0 & -2 & -2 & -4 & -3/2 & -6 \\ 0 & 4 & 4 & 8 & 2 & 4 \end{bmatrix} \qquad \_\_\_\_\_\_\_\_\_\_ \to R_2$$

Step 4: Cover the top row and look at the remaining (two) rows for the left-most nonzero column. Repeat Steps 1–3

$$\begin{bmatrix} 1 & 1 & 3 & 5 & 1/2 & 4 \\ 0 & -2 & -2 & -4 & -3/2 & -6 \\ 0 & 4 & 4 & 8 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 5 & 1/2 & 4 \\ 0 & -2 & -2 & -4 & -3/2 & -6 \\ 0 & 0 & & & & \end{bmatrix}$$

$R_3 + 2R_2 \to R_3$

The matrix is now in EF. Give another row equivalent EF.

Step 5. Final step to create the reduced echelon form: Beginning with the rightmost leading entry, and working upwards to the left, create 0's above each leading entry and scale rows to transform each leading entry into 1.

$$\begin{bmatrix} 1 & 1 & 3 & 5 & 1/2 & 4 \\ 0 & -2 & -2 & -4 & -3/2 & -6 \\ 0 & 0 & 0 & 0 & -1 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 5 & 1/2 & 4 \\ 0 & -2 & -2 & -4 & -3/2 & -6 \\ 0 & 0 & 0 & 0 & & \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 5 & 1/2 & 4 \\ 0 & -2 & -2 & -4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 8 \end{bmatrix}$$

$-R_3 \to R_3$
$R_2 + \frac{3}{2}R_3 \to R_2$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 5 & & \\ 0 & -2 & -2 & -4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 5 & 0 & 0 \\ 0 & & & & & \\ 0 & 0 & 0 & 0 & 1 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & & & & \\ 0 & 1 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 8 \end{bmatrix}$$

$R_1 - \frac{1}{2}R_3 \to R_1$
$-\frac{1}{2}R_2 \to R_2$
$R_1 - R_2 \to R_1$

The matrix is now in RREF.

Once a system is in RREF, we can read off the solution set. There are two types of variables in a solution set.

- **basic variable**: any *variable* that corresponds to a pivot column in the augmented matrix of a system.
- **free variable**: all nonbasic variables.

EXAMPLE 1.1.5. Here's the matrix in RREF. What is the corresponding system? What are the basic and free variables?

$$\begin{bmatrix} 1 & 0 & 2 & 3 & 0 & 3 \\ 0 & 1 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 8 \end{bmatrix} \quad \text{System:}$$

Notice that the system is *consistent*. There is no leading 1 (pivot) in the final column of the RREF of the augmented matrix That is there is no row of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

that would correspond to the inconsistent equation $0 = 1$.

The pivots are in columns 1, 2, and 5 so the basic variables are x_1, x_2, and x_5. The free variables are x_3 and x_4. Now we can actually solve for each of the basic variables in terms of the free variables.

$$\begin{aligned} x_1 &= -2x_3 - 3x_4 + 3 \\ x_2 &= \\ x_3 &= \\ x_4 &= \text{free} \\ x_5 &= \end{aligned}$$

In this example the system is consistent, but the solution set is infinite. We are free to choose any values for x_3 and x_4. Once those are selected, we get a particular solution for the entire system. For example, if $x_3 = 1$ and $x_4 = 0$, we get $(1, -4, 1, 0, 8)$ as a solution. If $x_3 = 0$ and $x_4 = 1$, we get $(0, -5, 0, 1, 8)$ as a solution. And so on.

Final Thoughts on Existence and Uniqueness of Solutions

Here are the EF's of three systems. What can we say about the existence and uniqueness of solutions in each case? (We could go to RREF, but that is not necessary to answer the question.)

$$A = \begin{bmatrix} 2 & 6 & 3 & 2 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 6 & 3 & 2 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 6 & 4 & 1 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

The system for A is consistent. The EF does not contain an equation of the form $0 = b$, where $b \neq 0$. That is, there is no pivot in the final column. The basic variables are x_1 and x_3, and the free variable is x_3. The system is consistent and has infinitely many solutions—one solution for each value of the free variable x_2.

The system for B is also consistent. The EF does not contain an equation of the form $0 = b$. That is, there is no pivot in the final column. This time x_1, x_2, and x_3 are basic since there are pivots in each of the first three columns. There are no free variables. The system is consistent and has only one solution. You can check that this solution is $(-2, 0, 2)$.

The system for C is inconsistent. The EF contains the equation $0 = 6$. That is, there is a pivot in the final column.

In any case we either have a pivot in the final column of the EF of the augmented matrix or not. If so, the system is not consistent. If not, if there are no free variables (pivots in all but the final column), the system has a unique solution. If the system is consistent and has a free variable, then it has infinitely many solutions. Thus, we have

THEOREM 1.1.6 (Existence and Uniqueness of Solutions)**.** A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the matrix does not have a row of the form

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}, \qquad \text{with } b \neq 0.$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

EXERCISE 1.1.7. Consider the following situations. Justify your answers.

(*a*) What is the largest possible number of pivots a 4×6 matrix can have? Why?

(*b*) What is the largest possible number of pivots a 6×4 matrix can have? Why?

(*c*) How many solutions does a consistent linear system of 3 equations and 4 unknowns have? Why?

(*d*) Suppose the *coefficient* matrix corresponding to a linear system is 4×6 and has 3 pivot columns. How many pivot columns does the *augmented* matrix have if the linear system is inconsistent?