Solutions to Homogeneous and Non-homogeneous Systems

FACT 1.1. Homogeneous Systems

(1) A homogenous system $A\mathbf{x} = \mathbf{0}$ is always consistent.

(2) A homogenous system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution if and only if the system has at least one free variable.

EXAMPLE 1.5.1. The homogenous system

$$\begin{aligned} 2x_1 + 4x_2 + x_3 - 2x_4 &= 0 \\ x_1 + 2x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_2 - 3x_4 &= 0 \end{aligned}$$

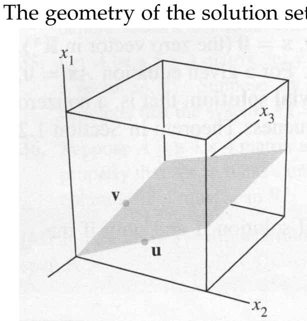
must have a non-trivial solution. (Why?)

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= -2x_2 + 3x_4 \\ x_2 &= \text{free} \\ x_3 &= -4x_4 \\ x_4 &= \text{free} \end{aligned}$$

The solutions in vector and parametric form:

$$\mathbf{x} \begin{bmatrix} -2x_2 + 3x_4 \\ x_2 \\ -4x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \end{bmatrix} = s\mathbf{u} + t\mathbf{v} \qquad (s, t \in \mathbb{R}).$$

The geometry of the solution set in $\mathbb{R}^4$.



The solution set is a plane through the origin, $\mathbf{0}$.

Another description:

$$\text{Span}\{\mathbf{u}, \mathbf{v}\} = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

Nonhomogeneous Systems

$A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} \neq \mathbf{0}$. By Theorem 2, such systems may or may not be consistent, depending on whether the final column contains a pivot.

EXAMPLE 1.5.2. Consider the nonhomogenous system

$$\begin{aligned} 2x_1 + 4x_2 + x_3 - 2x_4 &= 5 \\ x_1 + 2x_2 + x_3 + x_4 &= 3 \\ x_1 + 2x_2 - 3x_4 &= 2 \end{aligned}$$

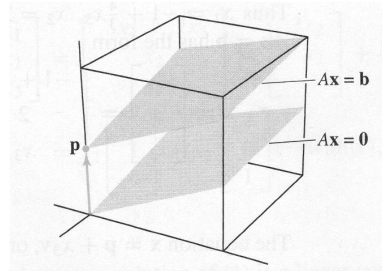
that has the same coefficient matrix as above. Reduce to solve:

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -3 & 2 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= 2 - 2x_2 + 3x_4 \\ x_2 &= \text{free} \\ x_3 &= 1 - 4x_4 \\ x_4 &= \text{free} \end{aligned}$$

The solutions in vector and parametric form:

$$\mathbf{x} \begin{bmatrix} 2 - 2x_2 + 3x_4 \\ x_2 \\ 1 - 4x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix} + x_2 \begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix} + x_4 \begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix} + s \begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix} + t \begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix} = \mathbf{p} + s\mathbf{u} + t\mathbf{v} \qquad (s, t \in \mathbb{R}).$$

Here $\mathbf{p}$ is a *particular solution* (check) to the system while $s\mathbf{u} + t\mathbf{v}$ gives the solutions to the companion homogeneous system. The effect is to translate or shift the entire solution set of the homogeneous system by a vector $\mathbf{p}$ to the solution set of the nonhomogeneous system. *To be able to do this, the $A\mathbf{x} = \mathbf{b}$ must have at least one particular solution $\mathbf{p}$—that is, the system must be consistent.* The following theorem shows that this is always the case.



THEOREM (Theorem 6). Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent and let $\mathbf{p}$ be a particular solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{w} = \mathbf{p} + \mathbf{v}_h,$$

where $\mathbf{v}_h$ is a solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Proof. Let $\mathbf{p}$ be a particular solution to the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ and let $\mathbf{w}$ be some other solution to $A\mathbf{x} = \mathbf{b}$. Let $\mathbf{v}_h = \mathbf{w} - \mathbf{p}$. Then

Goal: Find a vector $\mathbf{v}_h$ so that $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$. Notice that this equation tells us the 'answer.' Solving we must have $\mathbf{v}_h = \mathbf{w} - \mathbf{p}$. But is this $\mathbf{v}_h$ a solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$?

$$A\mathbf{v}_h = A(\mathbf{w} - \mathbf{p}) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad},$$

So $\mathbf{v}_h$ is a solution to the ____.
Morevover,

$$\mathbf{p} + \mathbf{v}_h = \underline{\qquad\qquad\qquad\qquad\qquad}$$

so $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ has the correct form. □

Summary. Writing the solutions to consistent system $A\mathbf{x} = \mathbf{b}$ in parametric form:

1. Row reduce $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ to reduced echelon form.
2. Express each basic variable in terms of the free variables.
3. Write the general solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors using the free variables as parameters.

EXERCISE 1.5.3. Suppose $\mathbf{y} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ is a solution to a 3×3 homogeneous system $A\mathbf{x} = \mathbf{0}$. Find a possible matrix A for this system. Hint: Think of $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$. Write the product $A\mathbf{y} = \mathbf{0}$ as a combination of the columns of A. What columns satisfy this relation?

Homework, Reading, Practice.

1. Due Wednesday, 10 February: Assignment 6, Part 1 online. Additional problems will be assigned for Friday.
2. WeBWorK HW3 will open on Tuesday. Due Monday, February 15th. These are problems on linear independence.
3. (*a*) Today: Dr. Caitlyn Parmelee from University of Nebraska-Lincoln. 4:30pm in Napier 201 (Snacks at 4:15pm). Title: Modeling in Mathematical Neuroscience
   (*b*) Thursday, Feb 11th: Dr. Alexander Diaz Lopez. 4:30pm in Napier 201 (Snacks at 4:15pm). Remember: Attendance at two talks is required.
4. Review Section 1.5. See previously assigned practice online.
5. (*a*) Read Section 1.7. Memorize the definition of linear independence. We will use this notion repeatedly.
   (*b*) Reading Check (not to be handed in, check answers in back of text): Section 1.7: Practice: Page 60 #1-4 (answers on page 62). Basics: Page 60-61: #1, 3(by inspection, avoid reduction), 5, 7(by inspection), 9 (nice), and 11.