THEOREM 0.0.1 (The Connections Theorem). Let A be an $n \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^n$ with standard matrix A. Then the following are equivalent.

- (a) A is non-singular
- (b) $A \sim I_n$
- (c) A has n pivots positions
- (d) A has pivot in every row
- (e) For any $\mathbf{b} \in \mathbb{R}^n$, the system $A\mathbf{x} = \mathbf{b}$ is consistent
- (f) Any $\mathbf{b} \in \mathbb{R}^n$ is a linear combination of the columns of A
- (g) The columns of A span $\mathbb{R}^n$
- (h) $T : \mathbb{R}^n \to \mathbb{R}^n$ is onto
- (i) A has pivot in every column (no free variables)
- (j) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$
- (k) The columns of A are independent
- (l) $T : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one
- (m) There is an $n \times n$ matrix C such that $CA = I_n$
- (n) There is an $n \times n$ matrix D such that $AD = I_n$
- (o) A^T is invertible
- (p) For any $\mathbf{b} \in \mathbb{R}^n$, the system $A\mathbf{x} = \mathbf{b}$ has a *unique* solution

Proof. $(a) \iff (n)$. $(a) \Rightarrow (n)$: Since A is invertible, let $D = A^{-1}$.

$(n) \Rightarrow (a)$: Given $n \times n$ matrices A and D such that $AD = I_n$. (Show A is invertible.) Since $AD = I$, this implies part (m) for the matrix D (i.e., substitute D for A and A for C in (m)). So D is invertible, so D^{-1} exists. Then

$$(AD)D^{-1} = ID^{-1} \text{ or } A = D^{-1}.$$

By Theorem 2.6, D^{-1} is invertible, so A is invertible (and $A^{-1} = D$). □

Proof. $(a) \iff (p)$. $(a) \Rightarrow (p)$: This is the First Connection Theorem (Theorem 2.5).

$(p) \Rightarrow (a)$: Given that for any $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ has a *unique* solution. Then for any $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ has *a* solution, which is part (e). So $(p) \Rightarrow (e) \Rightarrow (a)$. □

THEOREM 0.0.2 (The Onto Dictionary). Let A be an $m \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. Then the following are equivalent.

- (a) A has m pivots positions
- (b) A has pivot in every row
- (c) For any $\mathbf{b} \in \mathbb{R}^m$, the system $A\mathbf{x} = \mathbf{b}$ is consistent
- (d) Any $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of A
- (e) The columns of A span $\mathbb{R}^m$
- (f) $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto

THEOREM 0.0.3 (The One-to-One Dictionary). Let A be an $m \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. Then the following are equivalent.

- (a) A has n pivots positions
- (b) A has pivot in every column (no free variables)
- (c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$
- (d) The columns of A are independent
- (e) $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one

Reading and Practice

Office Hour Help: M & W 2:30–4:00, Tu 2:00–3:30, & F 1:30–2:30 or by appointment. Website: `http://math.hws.edu/~mitchell/Math204S16/index.php`.

1. Finish Assignment 11. A few NEW problems have been added.

2. Read/Reread Section 3.1 on Determinants. Review the Connections Theorem (Invertible Matrix Theorem) in Section 2.3. Our version is longer.

(*a*) Do the Practice Problems at the bottom of page 114. Then try: Page 115–116 #1, 3 (use A^T!), 5, 11, 13, 15, 17 (Hint: Use Theorem 8 and Theorem 6), 21 (like a question on a test), 23, 27(use the definition of invertible), 31 (use a 'dictionary').

(*b*) Determinant Computations: Pages 167ff #1, 3, 5, 11, 19, 23, 25, 29, 33.

Review Question

This question reviews several ideas. I will probably assign it Monday. JUST DO IT. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Let $\mathbf{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$. Assume that $T(\mathbf{u}) = T\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ and $T(\mathbf{v}) = T\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(*c*) Determine $T(3\mathbf{u} - 2\mathbf{v})$.

(*d*) Determine $T(\mathbf{e}_1)$. Hint: Express $\mathbf{e}_1$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Then proceed as in part (a).

(*e*) Similarly determine $T(\mathbf{e}_2)$.

(*f*) Determine the standard matrix A for T.

(*g*) Is A invertible?

(*h*) Determine all the vectors $\mathbf{w}$ such that $T(\mathbf{w}) = \mathbf{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

(*i*) Determine $T(\mathbf{x}) = T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(*j*) Is T onto? Explain carefully citing appropriate theorems.

(*k*) Is T one-to-one? Explain carefully citing appropriate theorems.