Office Hour Help: M & W 2:30–4:00, Tu 2:00–3:30, & F 1:30–2:30 or by appointment. Website: http://math.hws.edu/~mitchell/Math204S16/index.php.

Reading and Practice

☞ See Assignment 13 due Wednesday on the back.

1. Review Exam 2 and the answers. See me as necessary.

2. Today we will finish discussing determinants with an application to area and volume.

(*a*) Read Section 3.3 (we will not cover Cramer's Rule, but you may wish to read about it, especially if you are in Economics or Physics.

(*b*) Review Theorems 3.9 and 3.10. Practice: Page 184 #19 (draw the figure and determine the vectors that will provide the area), 21, 23, 27.

3. Next time, we will discuss general **Vector Spaces**. Read **Section 4.1**. We will focus on the axioms of a vector space and and subspaces. Close reading: Try pages 195–196 #1–3 (these are about closure). Come in with questions.

Recent Results about Determinants

THEOREM 4.1.1 (Determinants and Products). If A and B are $n \times n$ matrices, then $\det(AB) = \det A \cdot \det B$.

COROLLARY 4.1.2. Let A be $n \times n$.

(*a*) If m is a positive integer, then $\det A^m = (\det A)^m$.

(*b*) If A is invertible, then $\det A^{-1} = \dfrac{1}{\det A} = (\det A)^{-1}$.

THEOREM 4.1.3. If A is $n \times n$, then $\det A^T = \det A$.

☞ This means we can do column operations on a matrix to simplify calculating the determinant.

THEOREM 4.1.4. Let A is $n \times n$ and let k be any scalar. Then $\det(kA)$ in terms of $k^n \det A$.

THEOREM 4.1.5 (Area and Volume). If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

THEOREM 4.1.6 (Transformed Area and Volume). Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by the 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$ determined by $\mathbf{u}$ and $\mathbf{v}$, then the area of the transformed parallelogram $T(S)$ determined by $T(\mathbf{u})$ and $T(\mathbf{v})$ is given by

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\} = |\det A| \cdot |\det[\mathbf{u}\,\mathbf{v}]|.$$

Similarly, for $T : \mathbb{R}^3 \to \mathbb{R}^3$ and the volume of a parallelepiped.

Assignment 13

Due Wednesday after break. Prove the following theorems.

1. Prove: If A is $n \times n$, then $\det A^T = \det A$. (This means we can do column operations on a matrix to simplify calculating the determinant.)
2. Prove: If A is $n \times n$ and invertible, then $\det A^{-1} = \dfrac{1}{\det A} = (\det A)^{-1}$.
3. Prove: If A is square matrix n is a positive integer, then $\det A^n = (\det A)^n$. If you know how to do induction, try it that way.
4. Let A is $n \times n$ and let k be any scalar. Find a formula for $\det(kA)$ in terms of $\det A$.
5. A $n \times n$ invertible matrix U is **orthogonal** if $U^{-1} = U^T$. Prove that if U is orthogonal, then $\det U = \pm 1$.
6. An $n \times n$ matrix A is said to be **conjugate** to the $n \times n$ matrix B if there is some invertible $n \times n$ matrix M so that $A = MBM^{-1}$. Prove: If A is conjugate to B, then $\det A = \det B$.
7. Page 176, #40.
8. **Sebastien's Theorem.** Consider these three **off-diagonal** matrices.

$$A = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & d & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 \\ 0 & f & 0 & 0 & 0 & 0 \\ g & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   In each case, is the determinant of each the product of the diagonal elements? Show your work, justify your answer. Hint: Use row operations.
   ☞ Bonus: Can you figure out for which size matrices Sebastien's Theorem holds?

——————— Added Monday ———————

9. Section 3.3 (Page 184–5) Exercises #20, 22, and 28.
10. Suppose that $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$. If $\det A = -3$, evaluate $\det \begin{bmatrix} a & b & c-2a & d \\ e & f & g-2e & h \\ i & j & k-2i & l \\ m & n & o-2m & p \end{bmatrix}$.
11. Bonus: Find a formula (using determinants) for the area of a triangle whose vertices are $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$. Explain why your formula is correct.