*Reading and Practice.* Today we discuss general Vector Spaces. Read **Section 4.1**. Next time we will discuss subspaces (see pp. 193–195). Try pages 195–196 #1–4 (these are about closure) and 25–30. After reading about subspaces try #5, 7, 9, 11. **Hand In problems on next pages** for next class.

*Vector Spaces.* The algebraic properties of vectors in $\mathbb{R}^n$ occur other mathematical systems. In general, we can think of **vector space** as a collection of objects that behave just as vectors do in $\mathbb{R}^n$ under appropriate operations of addition and scalar multiplication.

**DEFINITION 4.1.1.** A **vector space** is a nonempty set $\mathbb{V}$ of elements called **vectors** on which are defined two operations called **addition** and **scalar multiplication** that satisfy the ten axioms below. (The axioms must hold for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{V}$ and for all scalars $c$ and $d$.)

1. $\mathbf{u}+\mathbf{v}$ is in _______________. (Closure)
2. $\mathbf{u}+\mathbf{v} =$ _______________. (______________________________)
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w} =$ _______________. (______________________________)
4. There is a vector (called the zero vector) $\mathbf{0} \in \mathbb{V}$ so that $\mathbf{u}+\mathbf{0} =$ _______________ for all $\mathbf{u} \in \mathbb{V}$. (Additive Identity)
5. For each $\mathbf{u} \in \mathbb{V}$, there is a vector $-\mathbf{u} \in \mathbb{V}$ so that $\mathbf{u}+(-\mathbf{u}) =$ _______________. (Additive Inverses)
6. $c\mathbf{u}$ is in _______________. (Closure)
7. $c(\mathbf{u}+\mathbf{v}) =$ _______________. (______________________________)
8. $(c+d)\mathbf{u} =$ _______________. (______________________________)
9. $(cd)\mathbf{u} =$ _______________.
10. $1\mathbf{u} =$ _______________. (Normalization)

*Example 1.* Let $\mathbb{V} = M_{22} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$. In this context $\mathbf{0} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$. And if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $-A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$. Most of the vector space axioms follow from Theorem 1 in Chapter 2. Which axioms still remain to be verified? Are they true (obvious)? So is $M_{22}$ is a vector space?

**THEOREM 2.1.1.** Let $A$, $B$, and $C$ be matrices of the same size and let $r$ and $s$ be scalars.

(1) $A+B = B+A$
(2) $(A+B)+C = A+(B+C)$
(3) $A+0 = A$, where 0 is the zero matrix
(4) $r(A+B) = rA+rB$
(5) $(r+s)A = rA+sA$
(6) $r(sA) = (rs)A$.

*Example 1′.* These same ideas apply to

$$\mathbb{V} = M_{mn} = \left\{ A = [a_{ij}] : A \text{ is an } m \times n \text{ matrix with } a_{ij} \in \mathbb{R} \right\}.$$

What is the additive identity matrix $\mathbf{0}$? What matrix is the additive inverse $-A$?

*Example 2.* $\mathbb{V} = \mathbb{R}^n$ with the usual operations of addition and scalar multiplication. The list of properties on page 27 ensures that $\mathbb{R}^n$ is a vector space.

ALGEBRAIC PROPERTIES OF $\mathbb{R}^n$: For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and scalars $c$ and $d$. Then $\mathbf{u}+\mathbf{v} \in \mathbb{R}^n$ and $c\mathbf{u} \in \mathbb{R}^n$. Further,

(1) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$  (2) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$
(3) $\mathbf{u}+\mathbf{0} = \mathbf{0}+\mathbf{u}$  (4) $\mathbf{u}+(-\mathbf{u}) = -\mathbf{u}+\mathbf{u} = \mathbf{0}$
(5) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$  (6) $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$
(7) $c(d\mathbf{u}) = (cd)\mathbf{u}$  (8) $1\mathbf{u} = \mathbf{u}$

Here $-\mathbf{u}$ denotes $(-1)\mathbf{u}$.

*Example 3.* This example will help you understand that vector spaces depend on both the set and how the operations of addition and scalar multiplication are defined. Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, the set of positive real numbers. Since the operations are not the usual ones we will use a different symbol. Define 'addition' as multiplication: $x \oplus y = xy$ and 'scalar multiplication' as raising to a power: $c \odot x = x^c$. Using these operations:

(1) Verify Axiom 1 $\mathbb{R}^+$ is closed under $\oplus$. Let $x, y \in \mathbb{R}^+$. Show $x \oplus y \in \mathbb{R}^+$.

VERIFICATION:

Always start with the appropriate set up. In this case: "Let $x, y \in \mathbb{R}^+ \ldots$"

(2) Verify Axiom 2: $x \oplus y = y \oplus x$.

(3) Verify Axiom 4; find the zero vector or additive identity $z$ so that $x \oplus z = x$ for all $x \in \mathbb{R}^+$.

(4) Verify Axiom 8; $(c + d) \odot (x) = (c \odot x) \oplus (d \odot x)$.

*Assignment 14A Due Friday. Name:* ___________________________

(5) Verify Axiom 6: $\mathbb{R}^+$ is closed under scalar multiplication, $c \odot x \in \mathbb{R}^+$.

(6) Verify Axiom 3: $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

(7) Verify Axiom 5; find the additive inverse $\ominus x$ so that $x \oplus (\ominus x) = 1$ where 1 is the identity that we found on the previous page.

(8) Verify Axiom 7: $c \odot (x \oplus y) = (c \odot x) \oplus (c \odot y)$.

(9) Verify Axiom 9: $(cd) \odot x = c \odot (d \odot x)$.

(10) Verify Axiom 10: $1 \odot x = x$.

*Example 4.* Let $\mathbb{V} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Define the operation of addition the usual way, but define scalar multiplication differently:

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix} \qquad\qquad r \odot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ra \\ b \end{bmatrix}.$$

Notice that both operations result in elements of $\mathbb{V}$ again so $\mathbb{V}$ is closed and Axioms 1 and 6 hold. Further, since addition is the usual operation of vector addition, we know that the Addition Axioms 2–5 also hold.

Determine whether $\mathbb{V}$ is a vector space by checking the remaining Multiplicative Axioms 7–10. Note: Use $r$ and $s$ as scalars not $c$ and $d$ as in the axioms since we have used $c$ and $d$ as components in the vectors.

*Additional Problems from the Text.* Do Page 197 Exercises #26 and 28. For each problem **copy the proof** and fill in the missing axiom.

**EXAMPLE:** Let $n \geq 0$ be an integer and let

$$\mathbf{P}_n = \text{the set of all polynomials of degree at most } n \geq 0.$$

Members of $\mathbf{P}_n$ have the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

where $a_0, a_1, \ldots, a_n$ are real numbers and $t$ is a real variable. The set $\mathbf{P}_n$ is a vector space:

Let $\mathbf{p}(t) = a_0 + a_1 t + \cdots + a_n t^n$ and $\mathbf{q}(t) = b_0 + b_1 t + \cdots + b_n t^n$. Let $c$ be a scalar.

***Axiom* 1*:***

The polynomial $\mathbf{p} + \mathbf{q}$ is defined as follows: $(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t)$. Therefore,

$$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t)$$

$$= (\underline{\hspace{3cm}}) + (\underline{\hspace{3cm}})t + \cdots + (\underline{\hspace{3cm}})t^n$$

which is also a ______________________ of degree at most ________. So $\mathbf{p} + \mathbf{q}$ is in $\mathbf{P}_n$.

***Axiom* 4*:***

$$\mathbf{0} = 0 + 0t + \cdots + 0t^n$$

(zero vector in $\mathbf{P}_n$)

$$(\mathbf{p} + \mathbf{0})(t) = \mathbf{p}(t) + \mathbf{0} = (a_0 + 0) + (a_1 + 0)t + \cdots + (a_n + 0)t^n$$

$$= a_0 + a_1 t + \cdots + a_n t^n = \mathbf{p}(t)$$

and so $\mathbf{p} + \mathbf{0} = \mathbf{p}$

***Axiom* 6*:***

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = (\underline{\hspace{2.5cm}}) + (\underline{\hspace{2.5cm}})t + \cdots + (\underline{\hspace{2.5cm}})t^n$$

which is in $\mathbf{P}_n$.

*Axiom 5* (additive inverse) If $\mathbf{p} = a_0 + a_1 t + \cdots + a_n t \in \mathbb{P}_n$, then $-\mathbf{p} =$ ______________________________ because

$$\mathbf{p} + (-\mathbf{p}) = (\qquad\qquad) + (\qquad\qquad)t + \cdots + (\qquad\qquad)t^n = \underline{\hspace{3cm}}.$$

*Axiom 2* (commutativity) If $\mathbf{p} = a_0 + a_1 t + \cdots + a_n t \in \mathbb{P}_n$ and $\mathbf{q} = b_0 + b_1 t + \cdots + b_n t \in \mathbb{P}_n$, then

$$\mathbf{p} + \mathbf{q} = (\qquad\qquad) + (\qquad\qquad)t + \cdots + (\qquad\qquad)t^n = \underline{\hspace{3cm}},$$

while

$$\mathbf{q} + \mathbf{p} = (\qquad\qquad) + (\qquad\qquad)t + \cdots + (\qquad\qquad)t^n = \underline{\hspace{3cm}}.$$

But addition of real numbers is commutative, so $a_i + b_i = b_i + a_i$, so $\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}$.

*Axiom 3* (associativity) Using the same idea, if $\mathbf{p}$ and $\mathbf{q}$ are as above, and $\mathbf{r} = c_0 + c_1 t + \cdots c_n t \in \mathbb{P}_n$, then $(\mathbf{p} + \mathbf{q}) + \mathbf{r} = \mathbf{p} + (\mathbf{q} + \mathbf{r})$ because

$$[(a_i + b_i) + c_i] = [\underline{\hspace{6cm}}].$$

*Axiom 7* (distributivity, vector addition) If $c$ is a scalar, then

$$\begin{aligned} c(\mathbf{p}+\mathbf{q}) &= c[(a_0+b_0)+(a_1+b_1)t+\cdots+(a_n+b_n)t^n] \\ &= (ca_0+cb_0)+(ca_1+cb_1)t+\cdots+(ca_n+cb_n)t^n \\ &= \underline{\hspace{3cm}}. \end{aligned}$$

*Axiom 8* (distributivity, scalar addition) If $c$ and $d$ are scalars, then

$$\begin{aligned} (c+d)\mathbf{p} &= (c+d)[a_0+a_1t+\cdots+a_nt^n] \\ &= \underline{\hspace{12cm}} \\ &= \underline{\hspace{3cm}}. \end{aligned}$$

*Axiom 9* (scalar multiplication)If $c$ and $d$ are scalars, then

$$\begin{aligned} c(d\mathbf{p}) &= c[da_0+da_1t+\cdots+da_nt^n] \\ &= \underline{\hspace{12cm}} \\ &= \underline{\hspace{3cm}}. \end{aligned}$$

*Axiom 10* (normalization) Is $1\mathbf{p}=\mathbf{p}$?

$$1\mathbf{p} = 1[a_0+a_1t+\cdots+a_nt^n] = \underline{\hspace{9cm}} = \underline{\hspace{3cm}}.$$

So all 10 axioms are satisfied, so $\mathbb{P}_n$ is a vector space.