Reading and Practice

Reread Section 4.1 and Read Section 4.2. The Null Space and Column Space are fundamental vector spaces associated with matrices and their corresponding transformations.

1. Practice Problems. Page 195ff [Check answers in the back.] #1, 3, 5, 7, 9, 11, 13, 15, 17, 21, 23, 25, 27, 29.

Today's Key Concepts

DEFINITION 4.1.1. A **subspace** of a vector space V is a subset H of $\mathbb{V}$ that has the following three properties.

(*a*) The zero vector of $\mathbb{V}$ is in H.

(*b*) H is closed under addition: That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

(*c*) H is closed under scalar multiplication: That is, for each $\mathbf{u} \in H$, and each scalar c, the vector $c\mathbf{u} \in H$.

THEOREM 4.1.2 (Subspaces are Vector Spaces)**.** Let H be a subspace of a vector space $\mathbb{V}$. Then H is, itself, a vector space.

THEOREM 4.1.3 (Spans are Subspaces)**.** If $\mathbf{v}_1, \ldots, v_p$ are vectors in a vector space $\mathbb{V}$, then Span $\{\mathbf{v}_1, \ldots, v_p\}$ is a subspace of $\mathbb{V}$.

NOTATION: We call Span $\{\mathbf{v}_1, \ldots, v_p\}$ the **subspace spanned** (or **generated**) by $\{\mathbf{v}_1, \ldots, v_p\}$. Given any subspace H of $\mathbb{V}$, a **spanning** or **generating set** for H is a set $\{\mathbf{v}_1, \ldots, v_p\}$ such that $H = \text{Span}\{\mathbf{v}_1, \ldots, v_p\}$.

DEFINITION 4.1.4. The **null space** of an $m \times n$ matrix A is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul}\, A = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

THEOREM 4.1.5 (Nul A is a Subspace)**.** The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Hand In Wednesday

1. (Review problem—see Test 2). Suppose that A and E are $n \times n$ matrices and E is an elementary. Prove: If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then $A^T \sim E$.

2. EXERCISE 4.1 #2: Let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$.

(-) Begin by giving an example of a vector $\mathbf{w} \in W$.

(*a*) If $\mathbf{u}$ is in W and c is any scalar, is $c\mathbf{u} \in W$? Why?

(*b*) Find $\mathbf{u}, \mathbf{v} \in W$ such that $\mathbf{u}+\mathbf{v} \notin W$.

3. For these three problems use the definition of a subspace on page 193.

(*a*) Page 196, Exercise #6: Let $H = \{\mathbf{p} \in \mathbb{P}_2 : \mathbf{p}(t) = a + t^2\}$. (1) Give an explicit example of a vector $\mathbf{q} \in H$. (2) Determine whether H a subspace of $\mathbb{P}_2$.

For part (a) the condition means that the coefficient of t^2 is 1.

(*b*) Page 196, Exercise #8: Let $H = \{\mathbf{p} \in \mathbb{P}_n : \mathbf{p}(0) = 0\}$. (1) Give an explicit example of a non-zero vector $\mathbf{q} \in H$. (2) Determine whether H a subspace of $\mathbb{P}_n$.

(*c*) Exercise #22, Page 196. Let F be a fixed 3×2 matrix. Determine whether $H = \{A \in M_{2\times4} : FA = \mathbf{0}_{3\times4}\}$ a subspace of $M_{2\times4}$.

4. Let $\mathbb{K}$ be the set of 2×2 **singular** matrices, i.e., $\mathbb{K} = \{A \in M_{2\times2} : \det A = 0\}$. Show that $\mathbb{K}$ is NOT a subspace of $M_{2\times2}$ by giving specific 2×2 matrices to show that one of the subspaces properties fails.

5. (*a*) Exercise #12, Page 196. Let $W = \left\{ \begin{bmatrix} 2s+4t \\ 2s \\ 2s-3t \\ 5t \end{bmatrix} \in \mathbb{R}^4 : s, t \text{ scalars} \right\}$. (1) Give an explicit example of a non-zero vector $\mathbf{v} \in W$. (2) Show that W is a subspace of $\mathbb{R}^4$.

(*b*) Exercise #14, Page 196. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 14 \end{bmatrix}$. Is $\mathbf{w}$ is the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(*c*) Exercise #16, Page 196. $W = \left\{ \begin{bmatrix} 1 \\ 3a-5b \\ 3b+2a \end{bmatrix} \in \mathbb{R}^3 : a, b \text{ scalars} \right\}$. Is W is a subspace of $\mathbb{R}^3$?

(*d*) Exercise #18, Page 196. $W = \left\{ \begin{bmatrix} 4a+3b \\ 0 \\ a+3b+c \\ 3b-2c \end{bmatrix} \in \mathbb{R}^4 : a, b, c \text{ scalars} \right\}$. Is W is a subspace of $\mathbb{R}^4$?

For Friday

6. Let $\mathbb{S}$ be the set of $n \times n$ **symmetric** matrices, i.e., $\mathbb{S} = \{A \in M_{n\times n} : A = A^T\}$. Determine whether $\mathbb{S}$ is a subspace of $M_{n\times n}$.

7. **Thinking ahead.** Now that we have defined general vector spaces we can generalize the idea of a linear transformation.

DEFINITION 4.1.6. If $\mathbb{V}$ and $\mathbb{W}$ are vector spaces, then $T : \mathbb{V} \to \mathbb{W}$ is a **linear transformation** if

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{V}$ and all scalars c.

DEFINITION 4.1.7. $T : \mathbb{V} \to \mathbb{W}$ is **onto** if for every $\mathbf{w} \in \mathbb{W}$, there is at least one vector $\mathbf{v} \in \mathbb{V}$ so that $T(\mathbf{v}) = \mathbf{w}$.

DEFINITION 4.1.8. $T : \mathbb{V} \to \mathbb{W}$ is **one-to-one** if whenever $T(\mathbf{u}) = T(\mathbf{v})$ then $\mathbf{u} = \mathbf{v}$.

(*a*) Let $T : \mathbb{R}^2 \to M_{2\times2}$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x & y \\ y & 2x \end{bmatrix}$. Prove T is a linear transformation.

(*b*) Determine whether T is one-to-one. (If $T(\mathbf{u}) = T(\mathbf{v})$ must $\mathbf{u} = \mathbf{v}$?)

(*c*) **Bonus**: Determine whether T is onto.

8. More problems will be added Wednesday.

A Shortcut for Determining Subspaces

THEOREM 1

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Proof: In order to verify this, check properties a, b and c of definition of a subspace.

a. **0** is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ since

$$\mathbf{0} = \_\_\_\_\mathbf{v}_1 + \_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\mathbf{v}_p$$

b. To show that Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ closed under vector addition, we choose two arbitrary vectors in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$:

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p$$

and

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$\mathbf{u} + \mathbf{v} = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p) + (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

$$= (\_\_\_\mathbf{v}_1 + \_\_\_\mathbf{v}_1) + (\_\_\_\mathbf{v}_2 + \_\_\_\mathbf{v}_2) + \cdots + (\_\_\_\mathbf{v}_p + \_\_\_\mathbf{v}_p)$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

So $\mathbf{u} + \mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

c. To show that Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ closed under scalar multiplication, choose an arbitrary number c and an arbitrary vector in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$:

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$c\mathbf{v} = c(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

$$= \_\_\_\_\mathbf{v}_1 + \_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\mathbf{v}_p$$

So $c\mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Since properties a, b and c hold, Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Recap

1. To show that H is a subspace of a vector space, use Theorem 1.
2. To show that a set is not a subspace of a vector space, provide a specific example showing that at least one of the axioms a, b or c (from the definition of a subspace) is violated.

4.2 Null Spaces, Column Spaces, & Linear Transformations

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbf{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\} \qquad \text{(set notation)}$$

THEOREM 2

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbf{R}^n$.

Proof: Nul A is a subset of $\mathbf{R}^n$ since A has n columns. Must verify properties a, b and c of the definition of a subspace.

Property (a) Show that **0** is in Nul A. Since _________, **0** is in __________.

Property (b) If **u** and **v** are in Nul A, show that $\mathbf{u} + \mathbf{v}$ is in Nul A. Since **u** and **v** are in Nul A,

__________ and __________.

Therefore

$$A(\mathbf{u} + \mathbf{v}) = \_\_\_\_\_\_ + \_\_\_\_\_\_ = \_\_\_\_\_\_ + \_\_\_\_\_\_ = \_\_\_\_.$$

Property (c) If **u** is in Nul A and c is a scalar, show that $c\mathbf{u}$ in Nul A:

$$A(c\mathbf{u}) = \_\_\_A(\mathbf{u}) = c\mathbf{0} = \mathbf{0}.$$

Since properties a, b and c hold, A is a subspace of $\mathbf{R}^n$.

Solving $A\mathbf{x} = \mathbf{0}$ yields an ***explicit description*** *of Nul* A.

EXAMPLE: Find an explicit description of Nul A where $A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$.

Solution: Row reduce augmented matrix corresponding to $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$