*Reading and Practice*

**1.** Re-read Section 4.2. Null spaces and column spaces are our first important uses of subspaces. We will deal with linear transformations on Friday. We will connect the concepts of Nul $A$ and Col $A$ to previous results. The blue box on page 230 is one connection. HOWEVER, if $A$ is a square ($n \times n$) think about ways to connect the Connections Theorem to Nul $A$ and Col $A$. *Make up your own new theorems.*

**2.** **5-minute Quiz on Friday** on key definitions so far in Chapter 4: vector space, subspace, null space and column space of matrix $A$, and the key theorems so far in Chapter 4: (1) Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of $\mathbb{V}$; (2) If $A$ is $m \times n$, then Nul $A$ is a subspace of $\mathbb{R}^n$. (3) If $A$ is $m \times n$, then Col $A$ is a subspace of $\mathbb{R}^m$.

**3.** Practice: Section 4.2. Page 205–206: #1, 3, 5, 7, 9, 11, 15, 17, 19, 21, 23, 25.

**4.** Coming up next: The **kernel of a linear transformation** $T$ (it turns out to be the same as the null space of the standard matrix of $T$) and the **range** of $T$ (which turns out to be the column space of $A$).

*Today's Key Concepts*

**DEFINITION 4.1.1.** The **null space** of an $m \times n$ matrix $A$ is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul}\, A = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

**THEOREM 4.1.2** (Nul $A$ is a Subspace)**.** The null space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^n$.

**DEFINITION 4.1.3.** The **column space** of an $m \times n$ matrix$A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ is the set of linear combinations of the columns of $A$. That is Col $A$ = Span $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.

**THEOREM 4.1.4** (Col $A$ is a Subspace)**.** If $A$ is an $m \times n$ matrix, the Col $A$ is a subspace of $\mathbb{R}^m$.

*Hand In Friday*

**1.** Let $\mathbb{S}$ be the set of $n \times n$ **symmetric** matrices, i.e., $\mathbb{S} = \{A \in M_{n\times n} : A = A^T\}$. Determine whether $\mathbb{S}$ is a subspace of $M_{n\times n}$.

**2.** **Thinking ahead.** Now that we have defined general vector spaces we can generalize the idea of a linear transformation.

**DEFINITION 4.1.5.** If $\mathbb{V}$ and $\mathbb{W}$ are vector spaces, then $T : \mathbb{V} \to \mathbb{W}$ is a **linear transformation** if

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{V}$ and all scalars $c$.

**DEFINITION 4.1.6.** $T : \mathbb{V} \to \mathbb{W}$ is **onto** if for every $\mathbf{w} \in \mathbb{W}$, there is at least one vector $\mathbf{v} \in \mathbb{V}$ so that $T(\mathbf{v}) = \mathbf{w}$.

**DEFINITION 4.1.7.** $T : \mathbb{V} \to \mathbb{W}$ is **one-to-one** if whenever $T(\mathbf{u}) = T(\mathbf{v})$ then $\mathbf{u} = \mathbf{v}$.

(*a*) Let $T : \mathbb{R}^2 \to M_{2\times 2}$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x & y \\ y & 2x \end{bmatrix}$. Prove $T$ is a linear transformation.

(*b*) Determine whether $T$ is one-to-one. (Assume $T(\mathbf{u}) = T(\mathbf{v})$. Must $\mathbf{u} = \mathbf{v}$?)

**3.** This is a useful fact: Page 197 #30. Hint: First do something to "get rid of" the scalar $c$ and use one of the properties from #26–29 on p. 197. Then use an Axiom.

Slow math. The point of this problem is to slow you down and make you realize how you are always using various vector space properties and axioms.

**4.** Let $\mathbb{V} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Define addition $\oplus$ in an unusual way and scalar multiplication as usual:

$$\begin{bmatrix} a \\ b \end{bmatrix} \oplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a^2 + c^2 \\ b + d \end{bmatrix} \qquad\qquad r\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ra \\ rb \end{bmatrix}.$$

Both operations result in elements of $\mathbb{V}$ so Axioms 1 and 6 hold. Carefully determine whether Axiom 3 holds.

**5.** Let $A$ be the matrix in Problem 3 on page 22 in Section 1.2. Give explicit descriptions of Nul $A$ and Col $A$, i.e., as spans of sets of vectors.

☞ Check your reduction in the back of the text.

**6.** These are simple checks on today's work. Justify each answer in one sentence.

(*a*) Page 205 #2

(*b*) Page 206 #10 (see Example 2)

(*c*) Page 206 # 16

(*d*) Page 206 #18

(*e*) Page 206 #24

**7.** Assume $A$ is $n \times n$. Prove: If Nul $A = \{\mathbf{0}\}$, then $\det A^T \neq 0$.

## *Hand In Monday*

**1.** Let $\mathbb{U} = \left\{ \begin{bmatrix} a+2 \\ 2a \\ b-a \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Determine whether $\mathbb{U}$ is a subspace of $\mathbb{R}^3$. Explain your answer.

**2.** We know that the set of all functions $\mathcal{F}$ defined on $(-\infty, \infty)$ is a vector space. Let $\mathbb{W} = \left\{ f \in \mathcal{F} : \int_0^{2\pi} f(x)\,dx = 0 \right\}$.

(*a*) List two familiar functions that are in $\mathbb{W}$.

(*b*) Determine whether $\mathbb{W}$ is a subspace of $\mathbb{F}$. You will need to use your Calculus 2 knowledge here.

**3.** (*a*) Again let $\mathcal{F}$ be the vector space of of all functions defined on $(-\infty, \infty)$. Let $\mathbb{C} = \{f \in \mathbb{F} : f \text{ is continuous}\}$. Give two examples of vectors in $\mathbb{C}$.

(*b*) Determine whether $\mathbb{C}$ is a subspace of $\mathcal{F}$. Use your Calculus 1 knowledge here. (E.g., Briggs and Cochran: Calculus text, section 2.6.)

**4.** Again let $\mathbb{C} = \{f \in \mathcal{F} : f \text{ is continuous}\}$. Let $T : \mathbb{C} \to \mathbb{R}$ by $T(f) = \int_0^1 f(x)\,dx$. Show that $T$ is a linear transformation. You will need to use your Calculus 2 knowledge again.

**5.** A few more will be added.

## 4.2 Null Spaces, Column Spaces, & Linear Transformations

The **null space** of an $m \times$ $n$ matrix $A$, written as Nul $A$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbf{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\} \qquad \text{(set notation)}$$

**THEOREM 2**

The null space of an $m \times n$ matrix $A$ is a subspace of $\mathbf{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of $m$ homogeneous linear equations in $n$ unknowns is a subspace of $\mathbf{R}^n$.

**Proof:** Nul $A$ is a subset of $\mathbf{R}^n$ since $A$ has $n$ columns. Must verify properties a, b and c of the definition of a subspace.

***Property (a)*** Show that **0** is in Nul $A$. Since __________, **0** is in __________.

***Property (b)*** If **u** and **v** are in Nul $A$, show that **u** + **v** is in Nul $A$. Since **u** and **v** are in Nul $A$,

__________ and __________.

Therefore

$$A(\mathbf{u}+\mathbf{v}) = \_\_\_\_\_\_\_ + \_\_\_\_\_\_\_ = \_\_\_\_\_\_ + \_\_\_\_\_\_ = \_\_\_\_.$$

***Property (c)*** If **u** is in Nul $A$ and $c$ is a scalar, show that $c\mathbf{u}$ in Nul $A$:

$$A(c\mathbf{u}) = \_\_\_A(\mathbf{u}) = c\mathbf{0} = \mathbf{0}.$$

Since properties a, b and c hold, $A$ is a subspace of $\mathbf{R}^n$.



Solving $A\mathbf{x} = \mathbf{0}$ yields an ***explicit description*** *of Nul $A$.*

**EXAMPLE:** Find an explicit description of Nul $A$ where $A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$.

*Solution:* Row reduce augmented matrix corresponding to $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} - \\ \\ \\ \\ \\ \end{bmatrix} + x_4 \begin{bmatrix} - \\ \\ \\ \\ \\ \end{bmatrix} + x_5 \begin{bmatrix} - \\ \\ \\ \\ \\ \end{bmatrix}$$

Then

$$\text{Nul } A = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$$

***Observations:***

1. Spanning set of Nul $A$, found using the method in the last example, is automatically linearly independent because of the positions of the 1's corresponding to the free variables:

$$c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = \_\_\_\_\_ \quad c_2 = \_\_\_\_\_ \quad c_3 = \_\_\_\_\_$$

2. If Nul A$\neq \{\mathbf{0}\}$, the the number of vectors in the spanning set for Nul $A$ equals the number of free variables in $A\mathbf{x} = \mathbf{0}$.

## The Contrast Between Nul $A$ and Col $A$

**EXAMPLE:** Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) The column space of $A$ is a subspace of $\mathbf{R}^k$ where $k =$_____.

(b) The null space of $A$ is a subspace of $\mathbf{R}^k$ where $k =$_____.

(c) Find a nonzero vector in Col $A$. (There are infinitely many possibilities.)

$$\_\_\_\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \_\_\_\begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix} + \_\_\_\begin{bmatrix} 3 \\ 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

(d) Find a nonzero vector in Nul $A$. Solve $A\mathbf{x} = \mathbf{0}$ and pick one solution.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 7 & 0 \\ 3 & 6 & 10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -2x_2$
$x_2$ is free
$x_3 = 0$

Let $x_2 =$ ___ and then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

***Contrast Between Nul $A$ and Col $A$ where $A$ is $m \times n$.***

*Col A is a subspace of* _______
*Nul A is a subspace of* _______

*In the example above: How would you check whether u = [1, 3, -1] is in Nul A?*

*How would you check whether v = [1, 2, 2, 1] in Col A?*