THEOREM 0.0.1 (The Onto Dictionary). Let A be an $m \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. Then the following are equivalent.

(a) A has m pivots positions

(b) A has pivot in every row

(c) For any $\mathbf{b} \in \mathbb{R}^m$, the system $A\mathbf{x} = \mathbf{b}$ is consistent

(d) Any $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of A

(e) The columns of A span $\mathbb{R}^m$

(f) ☞ $\operatorname{Col} A = \mathbb{R}^m$

(g) $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto

THEOREM 0.0.2 (The One-to-One Dictionary). Let A be an $m \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. Then the following are equivalent.

(a) A has n pivots positions

(b) A has pivot in every column (no free variables)

(c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$

(d) ☞ $\operatorname{Nul} A = \{\mathbf{0}\}$

(e) The columns of A are independent

(f) $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one

THEOREM 0.0.3 (The Connections Theorem). Let A be an $n \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^n$ with standard matrix A. Then the following are equivalent.

(a) A is non-singular

(b) $A \sim I_n$

(c) A has n pivots positions

(d) A has pivot in every row

(e) For any $\mathbf{b} \in \mathbb{R}^n$, the system $A\mathbf{x} = \mathbf{b}$ is consistent

(f) Any $\mathbf{b} \in \mathbb{R}^n$ is a linear combination of the columns of A

(g) The columns of A span $\mathbb{R}^n$

(h) $T : \mathbb{R}^n \to \mathbb{R}^n$ is onto

(i) A has pivot in every column (no free variables)

(j) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$

(k) The columns of A are independent

(l) $T : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one

(m) There is an $n \times n$ matrix C such that $CA = I_n$

(n) There is an $n \times n$ matrix D such that $AD = I_n$

(o) A^T is invertible

(p) For any $\mathbf{b} \in \mathbb{R}^n$, the system $A\mathbf{x} = \mathbf{b}$ has a *unique* solution

(q) $\det A \neq 0$

(r)

(s)

Reading and Practice

1. Practice: Section 4.2. Page 205–207: #1, 3, 5, 7, 9, 11, 15, 17, 19, 21, 23, 25. These problems are great and a bit harder: Page 235 #31, 33, and 35 (Challenge! XC).

2. Coming up next: Read Section 4.3 on Bases.

Hand In Monday

0. Remember the WeBWorK problem set due Tuesday

1. Let $\mathbb{U} = \left\{ \begin{bmatrix} a+2 \\ 2a \\ b-a \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Determine whether $\mathbb{U}$ is a subspace of $\mathbb{R}^3$. Be sure to justify your answer.

2. Assume A is $n \times n$. Prove: If Nul $A = \{\mathbf{0}\}$, then $\det A^T \neq 0$.

3. Background: Let $\mathcal{F}$ be the vector space of of all functions defined on $(-\infty, \infty)$. Let $\mathcal{C} = \{f \in \mathbb{F} : f \text{ is continuous}\}$. We can check that $\mathcal{C}$ is a subspace of $\mathcal{F}$. The zero function $\mathbf{0}(t) = 0$ is continuous so it is in $\mathcal{C}$. If f and g are both in $\mathcal{C}$, then both are continuous, so from Calculus I their sum $f+g$ is also continuous. So $f + g \in \mathcal{C}$. If c is a scalar and f is in $\mathcal{C}$, then f is continuous and from Calculus I, cf is also continuous. So $cf \in \mathcal{C}$. So $\mathcal{C}$ is a subspace of $\mathcal{F}$. Now here's the problem:

(*a*) Let $T : \mathcal{C} \to \mathbb{R}$ by $T(f) = \int_0^1 f(x)\, dx$. Show that T is a linear transformation. You will need to use your Calculus II knowledge.

(*b*) Bonus: Find an example of a function f in $\ker T$ such that f is not the zero function.

4. (*a*) Let $T : M_{2\times 2} \to \mathbb{R}^2$ by $T(A) = T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-c \\ b+d \end{bmatrix}$. Is T a linear transformation? Prove your result.

(*b*) Describe the form of the matrices in $\ker T$ in terms of their component entries. In other words,

$$\ker T = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \text{ some condition(s) involving } a, b, c, d \right\}.$$

Bonus if you can write them as the span of a set of vectors (matrices) in $M_{2\times 2}$.

5. (*a*) Complete the following: If A is $n \times n$, then $\det(cA) =$ ______________

(*b*) Let $T : M_{2\times 2} \to \mathbb{R}$ by $T(A) = \det(A)$. Determine whether T is a linear transformation. Hint: In light of part (a), which property of a linear transformation should you check first.

6. Bonus. Let H and K be subspaces of a vector space $\mathbb{V}$. The **intersection** of H and K, written as $H \cap K$, the set of vectors $\mathbf{v}$ in $\mathbb{V}$ that belong to both H and K. Prove that $H \cap K$ is a subspace of $\mathbb{V}$. [You will need to use the fact that H and K are both subspaces of $\mathbb{V}$ to verify the three subspace conditions.]

7. EZ Small Extra Credit. This problem is fairly easy. But if you need practice with subspace proofs, here's one more. Since this is a straightforward bonus problem, it should be done carefully and perfectly. Let $\mathbf{x}$ be some fixed (but unknown) vector in $\mathbb{R}^n$. Let

$$H = \{A \in M_{m\times n} : A\mathbf{x} = \mathbf{0}\}.$$

Is H a subspace of $M_{m\times n}$?

______________________________Coming Next Week ______________________________

8. (*a*) Let $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^T$. Is T a linear transformation? Prove your result carefully.

(*b*) Describe the form of the matrices in $\ker T$ in terms of their component entries.

$$\ker T = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \text{ some condition(s) involving } a, b, c, d \right\}.$$