4.3 Linearly Independent Sets; Bases

Definition

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $c_1 = 0, \ldots, c_p = 0$.

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists weights $c_1, \ldots, c_p$, not all 0, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

The following results from Section 1.7 are still true for more general vectors spaces.

A set containing the zero vector is linearly dependent.

A set of two vectors is linearly dependent if and only if one is a multiple of the other.

EXAMPLE: $\left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 3 & 0 \end{bmatrix} \right\}$ is a linearly __________________ set.

EXAMPLE: $\left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 9 & 11 \end{bmatrix} \right\}$ is a linearly __________________ set since

Theorem 4

An indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some vector $\mathbf{v}_j$ $(j > 1)$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

EXAMPLE: Let $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ be a set of vectors in $\mathbf{P}_2$ where $\mathbf{p}_1(t) = t$, $\mathbf{p}_2(t) = t^2$, and $\mathbf{p}_3(t) = 4t + 2t^2$. Is this a linearly dependent set?

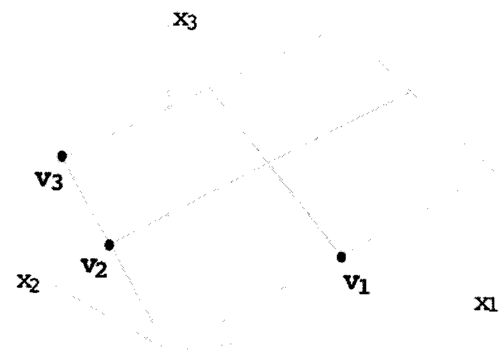
Solution: Since $\mathbf{p}_3 =$, $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a linearly __________________ set.

A Basis Set

Let H be the plane illustrated below. Which of the following are valid descriptions of H?

(a) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ (b) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$

(c) $H = \text{Span}\{\mathbf{v}_2, \mathbf{v}_3\}$ (d) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$



A *basis set* is an "efficient" spanning set containing no unnecessary vectors. In this case, we would consider the linearly independent sets $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_3\}$ to both be examples of basis sets or bases (plural for basis) for H.

DEFINITION

Let H be a subspace of a vector space V. An indexed set of vectors $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a basis for H if

(i) β is a linearly independent set, and

(ii) $H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$.

EXAMPLE: Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Show that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbf{R}^3$. The set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is called a **standard basis** for $\mathbf{R}^3$.

Solutions: Review the IMT Let

$A = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Since A has 3 pivots,