Office Hour Help: M & W 2:30–4:00, Tu 2:00–3:30, & F 1:30–2:30 or by appointment. Website: `http://math.hws.edu/~mitchell/Math204S16/index.php`.

Reading and Practice

1. (*a*) Review Section 4.3 and begin Section 4.4. This is more difficult material.

(*b*) Recent key definitions: Linear transformation between two general vector spaces, kernel and range of a linear transformation, **basis** for a vector space. Study that definition on page 209, but here's my variation:

DEFINITION 4.3.100. Let $\mathbb{V}$ be a vector space. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ is a **basis** for $\mathbb{V}$ if

1. $\mathcal{B}$ is linearly independent
2. $\mathcal{B}$ spans $\mathbb{V}$, i.e., $\mathbb{V} = \text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$.

(*c*) You should be to provide the proofs of the following facts that we did in class the last few days. You can find the outlines of two of them on the back of this sheet.

THEOREM 4.3.101 (Three Kernel Facts). Let $T : V \to W$ be a linear transformation between vector spaces. (1) $T(\mathbf{0}_V) = \mathbf{0}_W$. That is, $\mathbf{0}_V \in \ker T$.

(2) $\ker T$ is a subspace of V. (Corresponds to Nul A for a matrix transformation.)

(3) T is one-to-one if and only if $\ker T = \{\mathbf{0}_V\}$.

2. Practice: Page 213ff #1, 3, 5, 7, 9, 11, 13, 15, 19, 21. Excellent problems #23 and 29.

Work to Submit

1. Assignment 15 on Monday's Handout is Due Friday

2. Work on WeBWorK LHW10. Due Sunday at Midnight. (4 Problems)

3. Work on WeBWorK LHW11. Due Tuesday at Midnight. (10 Problems)

In Class Example

1. (*a*) Find a basis for the subspace of $\mathbb{R}^4$: Span$\left\{ \begin{bmatrix} 1\\1\\1\\0 \end{bmatrix} \begin{bmatrix} 2\\1\\1\\1 \end{bmatrix} \begin{bmatrix} 2\\2\\2\\0 \end{bmatrix} \begin{bmatrix} 1\\1\\1\\1 \end{bmatrix} \begin{bmatrix} 1\\0\\0\\2 \end{bmatrix} \right\}$

(*b*) Find a basis for the subspace of $M_{2\times 2}$: Span$\left\{ \begin{bmatrix} 1 & 1\\1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1\\1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2\\2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1\\1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0\\0 & 2 \end{bmatrix} \right\}$

(*c*) Find a basis for the subspace of $\mathbb{P}_3$: Span$\{1+t+t^2, 2+t+t^2+t^3, 2+2t+2t^2, 1+t+t^2+t^3, 1+2t^3\}$

(*d*) Can you generalize what is common to all these examples? Think about this as you read Section 4.4.

A couple of facts about linear transformations

Fact 1: Let $T : V \to W$ be a linear transformation between vector spaces. Then $T(\mathbf{0}_V) = \mathbf{0}_W$. That is, $\mathbf{0}_V \in \ker T$.

Proof: Since $0 \cdot \mathbf{0}_V = \mathbf{0}_V$,

$$T(\mathbf{0}_V) = T(0 \cdot \mathbf{0}_V) = \underline{\hspace{4cm}} = \underline{\hspace{4cm}}.$$

Fact 2: Let $T : V \to W$ be a linear transformation between vector spaces. Then T is one-to-one if and only if $\ker T = \{\mathbf{0}_V\}$.

Proof: Part 1: Suppose that T is one-to-one. By definition of one-to-one, there is ____________________ one vector $\mathbf{v} \in V$ so that $T(\mathbf{v}) = \mathbf{0}_W$. But Fact 1 says: ______________ $= \mathbf{0}_W$. So ______________ is the only vector in V mapped to $\mathbf{0}_W$. By definition of kernel, that means $\ker T =$ ______________.

Part 2: Suppose that $\ker T = \{\mathbf{0}_V\}$. Show that T is one-to-one. That means: Suppose that vectors $\mathbf{u}, \mathbf{v} \in V$ so that $T(\mathbf{u}) = T(\mathbf{v})$. Show $\mathbf{u} = \mathbf{v}$. But $T(\mathbf{u}) = T(\mathbf{v})$ so

$$T(\mathbf{u} - \mathbf{v}) = \underline{\hspace{4cm}}$$

Therefore,

$$T(\mathbf{u} - \mathbf{v}) = \underline{\hspace{4cm}} = \underline{\hspace{4cm}}.$$

So

$$\mathbf{u} - \mathbf{v} \in \underline{\hspace{3cm}} = \{\mathbf{0}_V\}$$

So $\mathbf{u} - \mathbf{v} =$ _______ and, therefore, $\mathbf{u} = \mathbf{v}$. By definition, T is one-to-one.