Math 204: Day 33

Read Section 4.5 which is on dimension. Read through the Class Work material on these sheets. We will probably not get to this today, but will next time. Try to work out the answers. Please prepare!

1. Review Section 4.4, especially change of coordinates which we discussed today. Read Section 4.5 which is on dimension.

a) **Test Monday**. It will cover Section 3.2 (Properties of Determinants) and Sections 4.1–4.5.

b) Practice in Section 4.4: Page 222ff #1, 5, 7, 13, 21, 23, and 24 (we did these in class, can you do them without looking at your notes?), 27, 31.

c) Once we get here: Practice in Section 4.5: Page 229ff #3, 5, 9, 11, 13, 15, 21, 23, 25. Now go back and find bases for Col A and Nul A in 13, 15, and 17.

2. Key Definitions (some will be on the exam): Linear transformation between two general vector spaces, kernel and range of a linear transformation, one-to-one, onto, basis for a vector space, subspace, null space of matrix A, and column space of matrix A. **New**: B-coordinates of $\mathbf{x}$, isomorphism, finite-dimensional vector space, $\dim V$.

Some practice problems for the exam on more recent material

1. a) Find a basis for the subspace of H of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$.

b) Find $\dim H$.

2. a) Let $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$. Show that B is a basis for $\mathbb{R}^3$.

b) If $\mathbf{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$, determine $[\mathbf{x}]_B$.

3. If $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$, find bases for Col A and Nul A and determine their respective dimensions.

4. Let $S = \left\{ \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 4 & 12 \\ 8 & 20 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}, \begin{bmatrix} -1 & 5 \\ 2 & 8 \end{bmatrix} \right\}$. Find a basis of Span S and find its dimension.

5. Page 229 #21. Is this already a homework problem?

6. Let $T : M_{2\times 2} \to \mathbb{R}^2$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+2d \\ b+c-d \end{bmatrix}$. Find a basis for $\ker T$ and determine $\dim \ker T$.

7. a) Show that the transformation $T : \mathbb{R}^2 \to \mathbb{P}_2$ by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (a+b) + at + (b-a)t^2$ is linear.

b) Show that it is one-to-one. (Hint: One method is to determine $\ker T$.)

Class work

Definition (Know): Suppose that $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V. The $\mathcal{B}$-**coordinates** of $\mathbf{x}$ are the weights $c_1, \ldots, c_n$ so that $\mathbf{x} = c_1\mathbf{b}_1 + \ldots, c\mathbf{b}_n$. Notation

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Definition (Know): A transformation $T : V \to W$ is an **isomorphism** if T is (1) linear, (2) one-to-one, and (3) onto.

Theorem 8

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for $\mathbb{V}$. Then the coordinate mapping $T : \mathbb{V} \to \mathbb{R}^n$ by $T(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$ is a linear transformation that is one-to-one and onto, i.e., T is an isomorphism.

Proof (Linear): To show that T is linear check the two properties. Take any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{V}$. Since $\mathcal{B}$ is a basis, we can write

$$\mathbf{u} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n \qquad \mathbf{v} = d_1\vec{b}_1 + d_2\vec{b}_2 + \cdots + d_n\vec{b}_n$$

So

$$\mathbf{u}+\mathbf{v} = (c_1+d_1)\vec{b}_1 + \cdots + (c_n+d_n)\vec{b}_n.$$

Then

$$T(\mathbf{u}+\mathbf{v}) = [\mathbf{u}+\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1+d_1 \\ \vdots \\ c_n+d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [u]_{\mathcal{B}} + [v]_{\mathcal{B}} = T(u)+T(v)$$

Next $r\mathbf{u} = r(c_1\vec{b}_1 + \cdots c_n\vec{b}_n) = rc_1\vec{b}_1 + \cdots + rc_n\vec{b}_n$. So

$$T(r\mathbf{u}) = [r\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} rc_1 \\ \vdots \\ rc_n \end{bmatrix} = \begin{bmatrix} \end{bmatrix} = r[\vec{u}]_{\mathcal{B}} = rT(\vec{u}).$$

Proof (One-to-one): To show that T is one-to-one we can use Fact 3 (Kernel): T is one-to-one if and only if $\ker T =$ ________. So let $\mathbf{u} \in \ker T$. Show that $\mathbf{u} = \vec{0}$. Since $\mathcal{B}$ is a basis, we can write

$$\mathbf{u} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n.$$

Since $\mathbf{u} \in \ker T$,

$$T(\mathbf{u}) = \vec{0}$$

$$[\vec{u}]_{\mathcal{B}} = \vec{0}$$

$$\begin{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

So each $c_i = 0$ and so $\mathbf{u} = 0\mathbf{b}_1 + \cdots + 0\mathbf{b}_n$, so $\mathbf{u} =$ ________.

arbitrary $\vec{x}$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

Proof (Onto): To show that T is onto, we take $\mathbf{x} \in \mathbb{R}^n$. Find $\mathbf{u} \in \mathbb{V}$ so that $T(\vec{u}) = \vec{x}$. This is the same as saying, find $\mathbf{u}$ so that

$$[\mathbf{u}]_{\mathcal{B}} = \mathbf{x} = \begin{bmatrix} \end{bmatrix}.$$

Which $\mathbf{u}$ will work? ____________________. Check $T(\mathbf{u}) = [\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} \end{bmatrix} =$ ____

EXAMPLE: Find a basis and the dimension of the subspace

$$W = \left\{ \begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} : a,b,c,d \text{ are real} \right\}.$$

Solution: Since $\begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} = a\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} + b\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} + c\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} + d\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$,

$W =$span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$

- $\mathbf{v}_1$ and $\mathbf{v}_2$ ____
- Note that $\mathbf{v}_3$ is a linear combination of ____________ by the Spanning Set Theorem, we may discard $\mathbf{v}_3$.
- **is** $\mathbf{v}_4$ a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. ________________ is a basis for W.
- Also, dim $W =$____.

EXAMPLE: *Dimensions of subspaces of $\mathbf{R}^3$*

0-dimensional subspace contains only the zero vector $\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$.

1-dimensional subspaces. Span$\{\mathbf{v}\}$ where $\mathbf{v} \neq \mathbf{0}$ is in $\mathbf{R}^3$.

These subspaces are _______________ through the origin.

2-dimensional subspaces. Span$\{\mathbf{u}, \mathbf{v}\}$ where $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbf{R}^3$ and are not multiples of each other.

These subspaces are _______________ through the origin.

3-dimensional subspaces. Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ where **u, v, w** are linearly independent vectors in $\mathbf{R}^3$. This subspace is $\mathbf{R}^3$ itself because the columns of $A = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$ span $\mathbf{R}^3$ according to the IMT.

The Spanning Theorem says if $\mathbb{H} = \mathrm{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, then we can discard (one at a time) any vector that is a linear combination of the others and still span H. Keep discarding until the remaining vectors are independent, thus producing a basis. The next theorem says we can go the 'other way', start with an **independent set and expand to a basis**.

Theorem 11

Let $\mathbb{H}$ be a subspace of a finite-dimensional vector space $\mathbb{V}$. Any linearly independent $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ in H can be expanded to a basis for H. Further, H is finite-dimensional and

$$\dim \mathbb{H} \leq \dim \mathbb{V}.$$

Proof: If we are lucky, Span $S = \mathbb{H}$. Then S is a _______________ for $\mathbb{H}$ because S is ___________________________.
Otherwise, if we are not lucky, S does not span H, so there's at least one vector $\mathbf{u}_{k+1} \in \mathbb{H}$ so that ___________________________________
But then $S_1 = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$ is an ____________________________ set because no vector is a linear combination of the preceding vectors (Theorem 4.4).

If S_1 does not span $\mathbb{H}$, expand again (and again) to get larger independent sets. Eventually the process must stop since no set of independent vectors in $\mathbb{V}$ can have more than _______________ vectors by Theorem 9. When the expansion of S stops say at S^*, all vectors in $\mathbb{H}$ are in span of S^* and hence this expanded set is a _______________ for $\mathbb{H}$.

EXAMPLE: Let $H = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Then H is a subspace of $\mathbf{R}^3$ and $\dim H < \dim \mathbf{R}^3$.

We could expand the spanning set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ to $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ to form a basis for $\mathbf{R}^3$.

Why do we know that this last set is a basis for $\mathbb{R}^3$?