Math 204: Day 34

1. Review Section 4.5 on dimension. Read Section 4.6 on Rank.

a) **Test Monday**. It will cover Section 3.2 (Properties of Determinants) and Sections 4.1–4.5.

b) Practice: Page 229 #3, 5, 9, 11, 13, 15, 21, 23, 25. Find bases for Col A and Nul A in 13 and 15.

2. Key Definitions (some will be on the exam): Linear transformation between two general vector spaces, kernel and range of a linear transformation, one-to-one, onto, **basis** for a vector space, subspace, null space of matrix A, and column space of matrix A. **New**: $\mathcal{B}$-coordinates of $\mathbf{x}$, isomorphism, finite-dimensional vector space, $\dim V$.

A few more practice problems for the exam

There were 8 problems on the last handout. These questions and answers are online. See website.

9. **a)** Determine whether $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T\left(\left[\begin{array}{c} a \\ b \end{array}\right]\right) = \left[\begin{array}{c} a+b \\ b^2 \end{array}\right]$ is a linear transformation.

b) Is $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \left[\begin{array}{c} \mathbf{p}(2) \\ 2\mathbf{p}(0) \end{array}\right]$ is a linear transformation.

c) If T is linear, find a basis for $\ker T$.

10. Is $\mathbb{W} = \left\{\left[\begin{array}{c} a \\ a^2 \end{array}\right] : a \in \mathbb{R}\right\}$ a subspace of $\mathbb{R}^2$?

11. Is $\mathbb{W} = \{\mathbf{p} \in \mathbb{P}_5 : \int_0^1 \mathbf{p}(t)\,dt = 1\}$ a subspace of $\mathbb{P}_5$?

12. Let $T : \mathbb{V} \to \mathbb{W}$ be a linear transformation. Prove that $\ker T$ is a subspace of $\mathbb{V}$. (We proved this in class, can you do the proof with looking it up?)

13. Page 196 #3.

14. Return to page 223 #32 (practice problem #3). If $\mathbf{p}(t) = 3 + t + 5t^2$ determine $[\mathbf{p}]_B$.

15. (Think!) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation that is one-to-one. What is the dimension of the Range of T? (Hint: Think about the standard matrix for this transformation.) What is the

16. (More thinking!) Suppose $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ are polynomials so that Span$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\} = \mathbb{H}$ *is a two-dimensional subspace* in $\mathbb{P}_5$. Describe how to find a basis for $\mathbb{H}$ by examining the four polynomials and doing almost no work.

Hand in Friday

Simple checks: Have you understood bases and dimension.

1. Page 229 #4, 12, 14, and 8 (like a null space problem).

EXAMPLE: Find the dimension of the subspace $W = \left\{ \begin{bmatrix} a+b+2c \\ a+2b+3c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$

SOLUTION: $\begin{bmatrix} a+b+2c \\ a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} = a \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} + b \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} + c \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} + d \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$

- $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly ______________________________
- $\mathbf{v}_3$ is a linear combination of _________. By the Spanning Set Theorem we may ___________ $\mathbf{v}_3$.
- Is $\mathbf{v}_4$ a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$?
- Conclusion: ____________________ is a basis for W and $\dim W =$ ___.

EXAMPLE: ***Dimensions of subspaces of $\mathbf{R}^3$***

0-*dimensional subspace* contains only the zero vector $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

1-*dimensional subspaces.* Span$\{\mathbf{v}\}$ where $\mathbf{v} \neq \mathbf{0}$ is in $\mathbf{R}^3$.

These subspaces are _______________ through the origin.

2-*dimensional subspaces.* Span$\{\mathbf{u}, \mathbf{v}\}$ where $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbf{R}^3$ and are not multiples of each other.

These subspaces are _______________ through the origin.

3-*dimensional subspaces.* Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent vectors in $\mathbf{R}^3$. This subspace is $\mathbf{R}^3$ itself because the columns of $A = [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}]$ span $\mathbf{R}^3$ according to the IMT.

EXAMPLE: Determine the dimensions of Nul A and Col A if

$$A \sim \begin{bmatrix} 1 & -2 & 5 & 0 & 1 \\ 0 & 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Spanning Theorem says if $\mathbb{H} = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, then we can discard (one at a time) any vector that is a linear combination of the others and still span $\mathbb{H}$. Keep discarding until the remaining vectors are independent, thus producing a basis. The next theorem says we can go the 'other way', start with an **independent set and expand to a basis**.

Theorem 11

Let $\mathbb{H}$ be a subspace of a finite-dimensional vector space $\mathbb{V}$ (say $\dim \mathbb{V} = n$). Any linearly independent $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ in $\mathbb{H}$ can be expanded to a basis for $\mathbb{H}$. Further, $\mathbb{H}$ is finite-dimensional and

$$\dim \mathbb{H} \leq \dim \mathbb{V}.$$

Proof: If we are lucky, Span $S = \mathbb{H}$. Then S is a __________ for $\mathbb{H}$ because S is already ______________________________.

Otherwise, if we are not lucky, S does not span $\mathbb{H}$, so there's at least one vector $\mathbf{u}_{k+1} \in \mathbb{H}$ so that ____. But then $S_1 = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$ is an ____________________ set because no vector is a linear combination of the preceding vectors (Theorem 4.4).

If S_1 does not span $\mathbb{H}$, expand again (and again) to get larger independent sets. Eventually the process must stop since no set of independent vectors in $\mathbb{V}$ can have more than __________ vectors by Theorem 9. When the expansion of S stops say at S^*, all vectors in $\mathbb{H}$ are in span of S^* and hence this expanded set is a __________ for $\mathbb{H}$.

EXAMPLE: Let $\mathbb{H} = \text{Span}\left\{\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}\right\}$. Then $\mathbb{H}$ is a subspace of $\mathbb{R}^4$. What vectors could we add to we expand that set $\left\{\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}\right\}$ to a basis for $\mathbb{R}^4$?

$$\left\{\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}, \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}\right\}$$

Why do you know that your set is a basis for $\mathbb{R}^4$?

EXAMPLE: Consider the set S of vectors five vectors in $\mathbb{P}_3$.

$$S = \{3 - 3t + 6t^2 + 3t^3, 5 - 2t + t^2, -4 + 4t - 8t^2 - 4t^3, 4 - t - t^2 - t^3, 2 + t - 5t^2 - 3t^3\}$$

Explain how you know that these vectors are dependent.

How would you determine the dimension of the subspace of $\mathbb{P}_3$ of spanned by S?