Math 204: Day 35

1. a) Read/Review Section 4.6. Try page 236 #1, 3, the next few use the Rank theorem: #5, 7, 9, 11, 13 and 15. Now try #19. Hint: What is dim Nul A? So what is the rank of A? #21 Hint: What must the rank be? So What is dim Nul A.

2. Skip ahead and Read Section 4.9 on Markov Chains, which will be our next topic.

3. Test Monday. It will cover Section 3.2 (Properties of Determinants) and Sections 4.1–4.5.

4. a) Key Definitions (some will be on the exam): Linear transformation between two general vector spaces, kernel and range of a linear transformation, one-to-one, onto, basis for a vector space, subspace, null space of matrix A, and column space of matrix A. $\mathcal{B}$-coordinates of $\mathbf{x}$, isomorphism, finite-dimensional vector space, $\dim V$. **New Today**: Row space of matrix A, rank A.

b) Key theorems: Theorems in Chapter 4: Theorems 4.1–12, especially Theorems 4.5 (Spanning Set Theorem), Theorems 4.9–12 about bases. You should be able to use all of these theorems and you should be able to fill in parts of the main theorems.

Homework Notes

1. When going from a spanning set to a basis by discarding vectors (using the Spanning Set Theorem), make sure to check that the remaining vectors are **independent**.

2. a) Notation. A basis is a set of of vectors, e.g., $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. This is not the same as Span $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, which is the set of all linear combinations of the vectors.

b) More notation. A coordinate vector is a vector in $\mathbb{R}^n$. So if $\mathbf{p} \in \mathbb{P}_2$, then using a basis $\mathcal{B}$, we might have $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$, not $1 + 4t$.

3. To go from the $\mathcal{B}$ basis to the standard basis use $P_{\mathcal{B}} = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_n]$

$$P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x} \text{ standard coordinates}$$

So to go from standard coordinates to $\mathcal{B}$ coordinates use

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}.$$

Class Work

1. If the null space of a 5×8 matrix A is 4-dimensional, what is $\dim \operatorname{Col} A$?

2. If the null space of a 8×6 matrix A is 4-dimensional, what is $\dim \operatorname{Row} A$?

3. If the null space of a 8×4 matrix A is 3-dimensional, what is $\dim \operatorname{Row} A$?

4. If A is 8×5, what is the largest possible rank? Smallest $\dim \operatorname{Nul} A$?

5. If A is 5×8, what is the largest possible rank? Smallest $\dim \operatorname{Nul} A$?

6. A homogeneous system $A\mathbf{x} = \mathbf{0}$ of 15 linear equations in 10 unknowns has three linearly independent solutions and all other solutions are linear combinations of these. Could this same set of solutions be described in fewer than 15 homogeneous equations? If so, what is the smallest number of such equations? $\dim \operatorname{Nul} A$? $\dim \operatorname{Col} A$? $\dim \operatorname{Row} A$?

7. Is it possible for a non-homogeneous system $A\mathbf{x} = \mathbf{b}$ of 8 equations in 6 unknowns to have a unique solution for some right-hand side $\mathbf{b}$? For **every** right-hand side? (Would would rank A have to be?)

Class work: Theorem 4.4 Review

A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p,\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ is a linear combination of the **preceding** vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Applying Theorem 4.4 to Row A

Find a basis for Row$A$ where

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 0 & 2 & 8 & 1 & 3 \\ 0 & 5 & 20 & 2 & 8 \end{bmatrix} \sim B = EF(A) = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 8 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \text{RREF}(A) = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. a) Row B is a subset of Row A because the rows of B are ____________________ the rows of A.

b) Likewise, Row B is a subset of Row A because row operations are ____________________.

c) So Row $A =$

2. a) By definition of row space, the non-zero rows of B ____________________ Row B.

b) We use Theorem 4.4 to show that the non-zero rows of B are independent. Look at what happens if we list the non-zero rows of B in reverse order:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 8 \\ 1 & 4 & 0 & 2 & -1 \end{bmatrix}$$

Now apply Theorem 4.4. No row can be a linear combination of the rows above it (i.e., rows **preceding** it) because there are always _______ above the pivot in this lower row because the pivots now move to the left. (This same observation is true for any matrix A and an echelon form B.)

c) Since the non-zero rows of B are ____________________ and ____________________, they are a ____________________ for Row B.

Theorem 4.13

a) If $A \sim B$, then Row A ____________________.

b) If $A \sim B$ and B is in echelon form, then the nonzero rows of B ____________________

and ____.

3. So what is the basis for Row A above and what is dim Row A? CAUTION!

Problem

Find bases the dimensions of the row space, column space, and null space of the matrix

$$A = \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & -1 & 5 & 2 & 3 \end{bmatrix}$$

Generalize your result: Suppose that A is an $m \times n$ matrix and that $B = \text{RREF}(A)$.

$$\dim \operatorname{Col} A = \# \text{ of } \underline{\hspace{4cm}} = \# \text{ of nonzero rows in } B = \dim \underline{\hspace{3cm}}$$

$$\dim \operatorname{Nul} A = \# \text{ of } \underline{\hspace{4cm}} = \# \text{ of } \underline{\hspace{4cm}} \text{columns in } A$$

Definition

The **Rank** of an $m \times n$ matrix A is the dimension of the column space of A.

Theorem 4.14 (The Rank Theorem)

Let A be an $m \times n$ matrix. Then

$$\operatorname{rank} A = \underline{\hspace{4cm}} = \underline{\hspace{4cm}} = \# \text{ of } \underline{\hspace{5cm}}$$

Further,

$$\operatorname{rank} A + \dim \operatorname{Nul} A = \underline{\hspace{2cm}}$$

Finally, since Row A = Col $\underline{\hspace{3cm}}$, then

$$\operatorname{rank} A = \operatorname{rank} \underline{\hspace{2cm}}$$

EXAMPLE: Suppose that a 5×8 matrix A has rank 5. Find $\dim$ Nul A, $\dim$ Row A and rank A^T. Is Col $A = \mathbf{R}^5$?

Solution:

$$\underbrace{\text{rank } A}_{\updownarrow \atop 5} + \underbrace{\dim \text{ Nul } A}_{\downarrow \atop ?} = \underbrace{n}_{\updownarrow \atop 8}$$

$$5 + \dim \text{ Nul } A = 8 \qquad \Rightarrow \qquad \dim \text{ Nul } A = \_\_\_\_\_$$

$$\dim \text{ Row } A = \text{rank } A = \_\_\_\_\_ \qquad \Rightarrow \qquad \text{rank } A^T = \text{rank } \_\_\_\_\_ = \_\_\_\_\_$$

Since rank A = # of pivots in $A = 5$, there is a pivot in every row. So the columns of A span $\mathbf{R}^5$ (by Theorem 4, page 43). Hence Col $A = \mathbf{R}^5$.

EXAMPLE: For a 9×12 matrix A, find the smallest possible value of $\dim$ Nul A.

Solution:

$$\text{rank } A + \dim \text{ Nul } A = 12$$

$$\dim \text{ Nul } A = 12 - \underbrace{\text{rank } A}_{\text{largest possible value}=\_\_\_\_\_}$$

smallest possible value of $\dim$ Nul A = _____

The Rank Theorem provides us with a powerful tool for determining information about a system of equations.

EXAMPLE: A scientist solves a homogeneous system of 50 equations in 54 variables and finds that exactly 4 of the unknowns are free variables. Can the scientist be *certain* that any associated nonhomogeneous system (with the same coefficients) has a solution?

Solution: Recall that

rank $A = \dim$ Col A = # of pivot columns of A

$\dim$ Nul A = # of free variables

In this case $A\mathbf{x} = \mathbf{0}$ of where A is 50×54.

By the rank theorem,

$$\text{rank } A + \_\_\_\_\_\_\_ = \_\_\_\_\_\_\_$$

or

$$\text{rank } A = \_\_\_\_\_\_\_.$$

Does $A\mathbf{x} = \mathbf{b}$ always have a solution?

Now try the Classwork Problems on page 1.