Math 204: Day 37

Office Hour Help: M & W 2:30–4:00, Tu 2:00–3:30, & F 1:30–2:30 or by appointment. Website: `http://math.hws.edu/~mitchell/Math204S16/index.php`.

1. Review Section 4.6. Read Section 4.9 on Markov Chains. Then skip to Section 5.1 on Eigenvectors.

   a) Practice—Review in Section 4.6. Page 236ff #3, 5, 7, 9, 11, 13, 15 and 17.

   b) Review. Key Terms from Section 4.6: Row space of A (Row A), the rank of A. Key results: the Rank Theorem (our version, see below), the **Connections (IMT) Extension**. I will let you cover this extension on your own.

   c) New. Key Terms from Section 4.9: probability vector, stochastic matrix, Markov chain, state vector, steady-state (equilibrium) vector, regular matrix. Key results: Theorem 4.18 (Regular Matrices and Steady State Vectors).

Theorem 4.14 (The Rank Theorem). Let A be an $m \times n$ matrix. Then

$$\operatorname{rank} A = \dim \operatorname{Col} A = \dim \operatorname{Col} A = \text{ \# of pivots}$$

Further,

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n$$

Finally, since Row $A =$ Col A^T, then $\operatorname{rank} A = \operatorname{rank} A^T$.

Hand In Wednesday: Assignment 18A

1. Page 236ff #4. Justify your answers: 6, 8, 10, 14, 16, 18, 20 (think RREF), and PROVE 28 (b).

Hand In Friday: Assignment 18B

Do problems 2 and 3 with a partner using `Maple` or other software. Hand in only one copy of your work. Remember to **use fractions** in `Maple` to obtain exact values. You should consult the `Maple` code for Day 37 at our website.

2. Page 261 #4 [Careful: Each column should have 3 entries, as should the probability vectors in (b,c)], 8, 10, 14, 16 (typical day means long run).

3. Consider a mouse in a maze consisting of four cells, as shown below. Assume that the mouse moves at random in the maze, in the following sense. If the mouse is currently in state j, it may move to either adjacent cell with equal probability. It never stays in its current cell.

4	3
1	2

   a) Determine the transition matrix P for this process.

   b) Is the transition matrix regular? Explain.

   c) Whether or not the transition matrix is regular, determine a steady-state vector for this system.

Markov Chains

Example 1. Suppose that a market research firms uses HWS students to study the soft drink preferences of college-age youths. In particular, they wish to study student preferences for Coke and Pepsi. (Coke and Pepsi are the two 'states' in the system.) At time $t = 0$ when the researchers first come to campus they survey the students and find that 60% of the students prefer Coke to Pepsi and 40% of the students prefer Pepsi to Coke. We indicate this with a vector

$$\mathbf{x}_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Coke} \\ \leftarrow \text{Pepsi} \end{array}$$

Over time, the researchers resurvey the students (say monthly) and obtain data vectors

$$\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \ldots$$

that represent the state of system (preferences) at times $t = 0, 1, 2, \ldots, n, \ldots$. In our example, the first entry will always represent the preference for Coke. Notice that the entries in each vector will sum to 1 since we are assuming that each student expresses a preference. A vector with nonnegative entries that sum to 1 is called a **probability vector**. Similarly a matrix P whose columns are probability vectors is called a **stochastic matrix**.

Definition. A **Markov chain** is a sequence of probability vectors, $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ along with a stochastic matrix P (your text often uses M) so that

$$\mathbf{x}_1 = P\mathbf{x}_0, \quad \mathbf{x}_2 = P\mathbf{x}_1 = P(P\mathbf{x}_0) = P^2\mathbf{x}_0, \quad \mathbf{x}_3 = P\mathbf{x}_2 = P(P^2\mathbf{x}_0) = P^3\mathbf{x}_0, \ldots$$

and more generally, for $n \geq 0$

$$\mathbf{x}_n = P\mathbf{x}_{n-1} = P^n\mathbf{x}_0.$$

The vectors $\mathbf{x}_n$ are usually referred to as the **state vectors** since they describe the state of the system at a particular time.

Example 1, continued. To make notation definite in our example, let

$$\mathbf{x}_n = \begin{bmatrix} c_n \\ p_n \end{bmatrix} \begin{array}{l} \leftarrow \text{Coke} \\ \leftarrow \text{Pepsi} \end{array}$$

be the percent Coke and Pepsi drinkers at time n. Suppose that after one month the researchers find that 90% of the Coke drinkers still prefer Coke, while 10% have switched to Pepsi. We can represent this vectorially (remembering which entry is for Coke, and which is for Pepsi) as

$$\begin{bmatrix} 0.9c_0 \\ 0.1c_0 \end{bmatrix} = c_0 \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix}.$$

Suppose 80% of the Pepsi drinkers still prefer Pepsi, while 20% have switched to Coke. We can represent this as

$$\begin{bmatrix} \quad p_0 \\ \quad p_0 \end{bmatrix} = p_0 \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

So at time 1, the state of the system is

$$\mathbf{x}_1 = \begin{bmatrix} c_1 \\ p_1 \end{bmatrix} = c_0 \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix} + p_0 \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} 0.9 & \quad \\ 0.1 & \quad \end{bmatrix} \begin{bmatrix} c_0 \\ p_0 \end{bmatrix} = P\mathbf{x}_0 = \begin{bmatrix} 0.9 & \quad \\ 0.1 & \quad \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix},$$

$$\text{where } P \text{ is the stochastic matrix } P = \begin{array}{c} \overbrace{\begin{array}{cc} \text{Coke} & \text{Pepsi} \end{array}} \\ \begin{bmatrix} 0.9 & \quad \\ 0.1 & \quad \end{bmatrix} \end{array} \left.\begin{array}{l} \text{Coke} \\ \text{Pepsi} \end{array}\right\}$$

The assumptions of a Markov chain model mean that the transition from $\mathbf{x}_1$ to $\mathbf{x}_2$ is the same as from $\mathbf{x}_0$ to $\mathbf{x}_1$. That is, after another month they again find that 90% of Coke drinkers still prefer Coke, while 80% of Pepsi drinkers stick with Pepsi. Then

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 0.62 \\ 0.38 \end{bmatrix} = \begin{bmatrix} 0.634 \\ 0.366 \end{bmatrix}.$$

We could also carry out this calculation by using the earlier formula $\mathbf{x}_2 = P^2\mathbf{x}_0$ and more generally $\mathbf{x}_n = P^n\mathbf{x}_0$. Using this, (and `Maple`) we find

$$\mathbf{x}_3 = P^3\mathbf{x}_0 = \begin{bmatrix} 0.6438 \\ 0.3562 \end{bmatrix}, \qquad \mathbf{x}_4 = P^4\mathbf{x}_0 = \begin{bmatrix} 0.65066 \\ 0.34934 \end{bmatrix}.$$

Ok, it looks like the number of Coke drinkers is increasing. Does everyone eventually prefer Coke? Or is there some upper limit to the number of Coke drinkers? What happens in the long run? Well, `Maple` helps here

$$\mathbf{x}_{50} = P^{50}\mathbf{x}_0 = \begin{bmatrix} 0.6666666655 \\ 0.3333333345 \end{bmatrix}, \qquad \mathbf{x}_{100} = P^{100}\mathbf{x}_0 = \begin{bmatrix} 0.6666666667 \\ 0.3333333333 \end{bmatrix}.$$

So what do you think happens in the long run?

$$\mathbf{q} = \lim_{n\to\infty} P^n\mathbf{x}_0 = \begin{bmatrix} \qquad\qquad \\ \\ \end{bmatrix}.$$

Interpret what this means.

What do we get if we multiply $\mathbf{q}$ by P?

$$P\mathbf{q} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \qquad\qquad \\ \\ \end{bmatrix} = \underline{\qquad\qquad}$$

Definition. If P is a stochastic matrix then a **steady-state vector** or **equilibrium vector** for P is a probability vector $\mathbf{q}$ so that $P\mathbf{q} = \mathbf{q}$.

Fact. Every stochastic matrix has a steady state vector.

Given the Fact above, is there a quick way to find the steady state vector without having to look at powers of P (or M) like we did in the example (can we avoid calculating P^n)? Yes! Look at the definition for a steady state vector. It says that $\mathbf{x}$ is a steady state vector if and only if $\mathbf{x}$ is a *probability vector* and

$$\begin{aligned} P\mathbf{x} = \mathbf{x} &\iff P\mathbf{x} - \mathbf{x} = \underline{\qquad\quad} \\ &\iff P\mathbf{x} - I\underline{\qquad\quad} = \mathbf{0} \\ &\iff (\underline{\qquad\quad})\mathbf{x} = \mathbf{0} \\ &\iff \mathbf{x} \in \operatorname{Nul}(P - I) \end{aligned}$$

Example 1, again. So we need to find a basis for the null space of $P - I$. Just do it.

$$P - I = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix} \sim \begin{bmatrix} \qquad\qquad \\ \\ \end{bmatrix} \sim \begin{bmatrix} \qquad\qquad \\ \\ \end{bmatrix}.$$

So a basis for $\operatorname{Nul}(P - I)$ is

$$\mathcal{B} = \left\{ \qquad\qquad \right\} = \{\mathbf{b}\}$$

Notice all solutions to $(P - I)\mathbf{x} = \mathbf{0}$ are multiples of $\mathbf{b}$. To get the steady-state vector $\mathbf{q}$ make $\mathbf{b}$ a probability vector. Divide $\mathbf{b}$ by the sum of its components. (Note: The sum of the absolute values of the components is called the **1-norm** of the vector and is denoted by $||\mathbf{b}||_1$.)

$\mathbf{q} =$

Is it possible to have two or more independent steady-state vectors? What would that say about $\dim(\mathrm{Nul}(P - I))$? There is a simple condition that guarantees that there is **unique** steady-state vector for a stochastic matrix P.

Definition: A stochastic matrix P is **regular** if some power P^k has all positive entries.

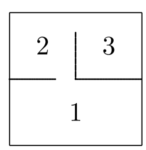
Question. Are the following stochastic matrices regular?

$$A^2 = \begin{bmatrix} 0.4 & 1 \\ & \end{bmatrix}\begin{bmatrix} 0.4 & 1 \\ & \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ & \end{bmatrix} \qquad B^2 = \begin{bmatrix} 1 & .4 \\ & \end{bmatrix}\begin{bmatrix} 1 & .4 \\ & \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ & \end{bmatrix}$$

Theorem 18 (Regular Matrices and Steady-State Vectors) If P is a regular stochastic matrix, then P has a **unique** steady-state vector $\mathbf{q}$. Further, if $\mathbf{x}_0$ is any initial state, then the Markov chain $\{\mathbf{x}_n : n = 0, 1, 2, \dots\}$ converges to $\mathbf{q}$, that is, $\lim_{n\to\infty} \mathbf{x}_n = \lim_{n\to\infty} P^n\mathbf{x}_0 = \mathbf{q}$. In fact,

$$\lim_{n\to\infty} P^n = [\mathbf{q}\,\mathbf{q}\,\dots\,\mathbf{q}].$$

Example 2. A mouse is placed in the maze below. During a fixed time interval it may choose to **move** to any connecting adjacent compartment or it may **stay** in the compartment it currently occupies, all choices being equally probable.



a) What is the transition (stochastic) matrix for this situation?

b) Show that this is a regular Markov process.

c) Determine the steady-state vector $\mathbf{q}$ for this process.

Note: Maple code for this example is available online at our website.

Example 3 (HW Problem). The weather in Columbus is either good, indifferent, or bad on any given day. If the weather is good today, there is a 40% chance it will be good tomorrow, a 30% chance that it will be indifferent, and a 30% chance it will be bad. If the weather is indifferent today, there is a 50% chance it will be good tomorrow and a 20% chance that it will be indifferent. If the weather is bad today, there is a 30% chance it will be good tomorrow and a 40% chance that it will be indifferent.

a) What is the transition (stochastic) matrix for this situation?

b) Suppose that there is a 50% of good weather for today and a 50% chance of indifferent weather. What are the chances of bad weather tomorrow?

c) Suppose that the predicted weather for Monday is 60% indifferent and a 40% chance bad weather. What are the chances of good weather on Wednesday?

d) In the long run, what is the probability of good weather on an average day?