Office Hour Help: M & W 2:30–4:00, Tu 2:00–3:30, & F 1:30–2:30 or by appointment. Website: http://math.hws.edu/~mitchell/Math204S16/index.php.

Practice and Reading

Try to do the proof on the LAST page. Volunteer to present in class next time.

1. (*a*) Review Sections 5.1 and 5.2 in your text. We should finish most of Section 5.2 today. I expect to begin Section 5.3 on Monday, so read ahead. Skim that.
2. (*a*) **Practice**. First try Page 273 #1, 3, 7, 9, 15. Now go back and try Page 271 #3, 7, 9, 13, 17. Now try Page 271 21, 23 (for 2×2 only). Now try Finally try #35. This shows the utility of a basis of eigenvectors.
   (*b*) Key Terms: eigenvalue, eigenvector, algebraic multiplicity, characteristic equation (polynomial), geometric multiplicity, and similarity.

Hand in Wednesday

☞ **Remember:** WeBWorK set SHW13. Due Monday night.

1. Find the characteristic polynomial and the real eigenvalues for the following four matrices. These show the four different possible results for 2×2 matrices. You may need to use the quadratic formula.

   (*a*) $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ (*b*) $\begin{bmatrix} 5 & -3 \\ -4 & 3 \end{bmatrix}$ (*c*) $\begin{bmatrix} 3 & -4 \\ 4 & 8 \end{bmatrix}$ (*d*) $\begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$
2. Find bases for the eigenspaces in #1(a) and #1(d).
3. Find the eigenvalues and bases for the eigenspaces for $A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.
4. List the eigenvalues and their multiplicities for $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}$.
5. Exercise #18 page 280. (You are trying to find h to make E_4 two-dimensional.)
6. Prove the following (see hint in margin):

THEOREM 4.3.100. Assume λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$. Then λ is an eigenvalue of A^T.

☞: Use the Characteristic Equation and properties of transposes and determinants.

7. Prove the following:

THEOREM 4.3.101. Assume A is invertible and λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$. Then λ^{-1} is an eigenvalue of A^{-1} corresponding to the same eigenvector $\mathbf{v}$.

☞: Start with $A\mathbf{v} = \lambda\mathbf{v}$. Use A^{-1}. Make sure to explain why λ^{-1} exists.

8. Prove the following:

THEOREM 4.3.102. Assume λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$. Then λ^2 is an eigenvalue of A^2 corresponding to the same eigenvector $\mathbf{v}$.

☞: Consider $A^2\mathbf{v} = A(A\mathbf{v})$.

BONUS: If you know induction, then prove: For any positive integer n, if λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then λ^n is an eigenvalue of A^n corresponding to the same eigenvector $\mathbf{v}$.

☞ OVER

9. Page 272 #22 (give a reference), 36.

10. (*a*) Use `Maple` to find the eigenvalues and eigenvectors for the **TRANSPOSE** of the matrix page 273, #39. (See the `Maple` sample code below.) According to problem 2, you should get the same eigenvalues as for A. Did you (look in the back of the text)? Are the eigenvectors the same?

with(*LinearAlgebra*) :
Find the eigenvalues and bases for the corresponding eigenspaces for

$$A := \begin{bmatrix} 9 & -56 & -14 & 42 \\ -4 & 32 & -14 & -33 \\ -2 & -28 & 6 & 21 \\ -4 & 44 & -14 & -45 \end{bmatrix}:$$

Get the characteristic polynomial using the determinant:

Determinant(*A* − lambda *IdentityMatrix*(4))

$$\lambda^4 - 2\lambda^3 - 311\lambda^2 + 312\lambda + 24336 \quad \textbf{(1)}$$

Now cut and paste the equation into the solve command:
solve(24336 + λ⁴ − 2 λ³ − 311 λ² + 312 λ = 0, lambda)

$$-12, -12, 13, 13 \quad \textbf{(2)}$$

We could have done this a bit more directly using the CharacteristicPolynomial command
CharacteristicPolynomial(*A*, lambda)

$$\lambda^4 - 2\lambda^3 - 311\lambda^2 + 312\lambda + 24336 \quad \textbf{(3)}$$

In fact we don't even need to compute the characteristic polynomial separately, you can embed it in the solve command

solve(*CharacteristicPolynomial*(*A*, lambda) = 0, lambda)

$$-12, -12, 13, 13 \quad \textbf{(4)}$$

To find the eigenspaces use the usual commands. For -12:
ReducedRowEchelonForm(*A* − (−12)*IdentityMatrix*(4))

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(5)}$$

from which you can get the basis. (What would it be?) Even easier, use
NullSpace(*A* − (−12)*IdentityMatrix*(4))

$$\left\{ \begin{bmatrix} 0 \\ \frac{3}{4} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\} \quad \textbf{(6)}$$

Class Work

THEOREM 4.3.103 (Eigenvectors corresponding to Distinct Eigenvalues are Independent). If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

Proof. (By Contraposition.) Assume $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is **dependent**. (Show $\lambda_1, \ldots, \lambda_r$ are not distinct. . . so at least two of them are equal).

By the definition of dependent, by Theorem 4.4, one of the vectors—say $\mathbf{v}_{p+1}$—is a non-trivial combination of the earlier vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_p$ which are independent. So

$$\underline{\hspace{8cm}} = \mathbf{v}_{p+1} \tag{4.1}$$

where some $c_i \neq$ _______

Multiply (**??**) by A:

$$\underline{\hspace{8cm}} = A\mathbf{v}_{p+1} \tag{4.2}$$

Since each $\mathbf{v}_k$ is an eigenvector corresponding to λ_k, we can rewrite (**??**) as

$$\underline{\hspace{8cm}} = \lambda_{p+1}\mathbf{v}_{p+1} \tag{4.3}$$

Also multiply (**??**) by λ_{p+1}:

$$\underline{\hspace{8cm}} = \lambda_{p+1}\mathbf{v}_{p+1} \tag{4.4}$$

Subtract: (**??**)−(**??**):

$$\underline{\hspace{8cm}} = \mathbf{0} \tag{4.5}$$

But $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are independent, so all the coefficients in (**??**) equal 0 so

$$c_1(\lambda_1 - \lambda_{p+1}) = 0,\ c_2(\lambda_2 - \lambda_{p+1}) = 0,\ \ldots,\ c_p(\lambda_p - \lambda_{p+1}) = 0$$

Since the combination in (**??**) at the outset was non-trivial, some $c_i \neq$ _______. So $c_i(\lambda_i - \lambda_{p+1}) = 0$ means

$$\lambda_i - \lambda_{p+1} = \underline{\hspace{2cm}} \qquad \text{which means } \underline{\hspace{3cm}}$$

So the eigenvalues are ____________________________. □

Class Work—The proof of The Diagonalization Theorem

Definition. An $n \times n$ matrix A is **diagonalizable** if A is similar to a diagonal matrix. (That is, $A = PDP^{-1}$, where D is diagonal. Note: This also means $P^{-1}AP = D$.)

The Diagonalization Theorem. An $n \times n$ matrix A is diagonalizable if and only if A has n independent eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. In fact, if $P = [\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n]$ and

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

is the diagonal matrix whose diagonal entries are the eigenvalues corresponding to $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then $A = PDP^{-1}$.

Proof. $\Rightarrow$. Given A is diagonalizable. So $A =$ ______________ where _______ is invertible and _______ is _____________________. So $AP =$ ______________. Let $P = [\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n]$. Then

$$AP = A[\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n] = [\qquad\qquad\qquad\qquad\qquad\qquad\qquad]$$

while, since D is diagonal with diagonal entries $\lambda_1, \ldots, \lambda_n$, then

$$PD = [\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\qquad\qquad\qquad\qquad\qquad\qquad\qquad]$$

But $AP = PD$ so $A\mathbf{v}_1 =$ ______________, so _______ is an ____________________________ corresponding to _______. Similarly for $\mathbf{v}_2 \ldots \mathbf{v}_n$. Since $P = [\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n]$ is invertible, $\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n$ are ____________________________by the IMT.

$\Leftarrow$ Given A has n independent eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ corresponding to $\lambda_1, \ldots, \lambda_n$ (λ's are not necessarily, distinct by the way). Let $P = [\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n]$. P is ____________________________(IMT). Let

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Then

$$AP = [\qquad\qquad\qquad\qquad\qquad\qquad\qquad] = [\lambda_1\mathbf{v}_1\, \ldots\, \lambda_n\mathbf{v}_n] = PD$$

because $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are ___________________________________. So

$$AP = PD \Rightarrow A = \underline{\qquad\qquad\qquad}$$

So A is ___________________________________.