Office Hour Help: M & W 2:30–4:00, Tu 2:00–3:30, & F 1:30–2:30 or by appointment. Website: `http://math.hws.edu/~mitchell/Math204S16/index.php`.

Practice and Reading

1. (*a*) Review Sections 5.2 and 5.3. We will discuss stage-matrix models in the final classes.

(*b*) Read the Case Study on Owl populations (handout). We will conclude the term with a study of **stage-matrix** models. Skim section 5.6 pages 301–middle of 303, 307(Owls)–309 in your text.

2. (*a*) **Practice**. New: Page 286ff: #1(see Example 2), 3, 5, 7, 9, 11, 17, 21.

(*b*) Previously suggested: **Practice**. First try Page 273 #1, 3, 7, 9, 15. Now go back and try Page 271 #3, 7, 9, 13, 17. Now try Page 271 21, 23 (for 2×2 only). Now try Finally try #35. This shows the utility of a basis of eigenvectors.

(*c*) Key Terms: eigenvalue, eigenvector, algebraic multiplicity, characteristic equation (polynomial), geometric multiplicity, similarity, diagonalizable, eigenvector basis for $\mathbb{R}^n$.

On the next assignment

1. Page 286–287: Just find P and D if possible: #8, 10, and 12. Note for #12 you are given the eigenvalues.

2. Here's an easy problem: Without doing any calculation give an eigenvalue of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$. Hint: Connections Extension to Eigenvalues.

3. Suppose that an $n \times n$ matrix A has eigenvector $\mathbf{v}_1$ corresponding to eigenvalue λ_1. On the last assignment, you proved that $A^2\mathbf{v}_1 = \lambda_1^2\mathbf{v}$.

(*a*) Assume that for some positive integer k, you know $A^k\mathbf{v}_1 = \lambda_1^k\mathbf{v}_1$. Prove that $A^{k+1}\mathbf{v}_1 = \lambda_1^{k+1}\mathbf{v}_1$. Hint: Think of A^{k+1} as AA^k.

COMMENT: Here's what the previous part means. Since we know that A has eigenvector $\mathbf{v}_1$ corresponding to eigenvalue λ_1, we know that $A^1\mathbf{v}_1 = \lambda_1^1\mathbf{v}_1$. By part (a) with $k = 1$, we now know that $A^{k+1}\mathbf{v}_1 = A^{1+1}\mathbf{v}_1 = A^2\mathbf{v}_1 = \lambda_1^2\mathbf{v}_1$. By part (a) again with $k = 2$, we now know that $A^{k+1}\mathbf{v}_1 = A^{2+1}\mathbf{v}_1 = A^3\mathbf{v}_1 = \lambda_1^3\mathbf{v}_1$. By part (a) with $k = 3$, we now know that $A^{k+1}\mathbf{v}_1 = A^{3+1}\mathbf{v}_1 = A^4\mathbf{v}_1 = \lambda_1^4\mathbf{v}_1$, and so on. So we have shown

THEOREM 4.3.2. Suppose that a square matrix A has eigenvector $\mathbf{v}_1$ corresponding to eigenvalue λ_1. Then for any positive integer n, $A^n\mathbf{v}_1 = \lambda_1^n\mathbf{v}_1$

(*b*) Suppose that an $n \times n$ matrix A has eigenvector $\mathbf{v}_1$ corresponding to λ_1. Let c be any scalar. Using the theorem above and basic matrix algebra, prove: For any positive integer n, $A^n(c\mathbf{v}_1) = c\lambda_1^n\mathbf{v}$. (This is a quick proof.)

(*c*) Ok, now go a step further. Suppose that an $n \times n$ matrix A has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$. If the vector $\mathbf{x}$ can be expressed as $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$, then use your result above and matrix algebra to show

$$A^n\mathbf{x} = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2 + \cdots + c_p\lambda_p^n\mathbf{v}_p$$

Class Work—The proof of The Diagonalization Theorem

Definition. An $n \times n$ matrix A is **diagonalizable** if A is similar to a diagonal matrix. (That is, $A = PDP^{-1}$, where D is diagonal. Note: This also means $P^{-1}AP = D$.)

The Diagonalization Theorem. An $n \times n$ matrix A is diagonalizable if and only if A has n independent eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. In fact, if $P = [\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n]$ and

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

is the diagonal matrix whose diagonal entries are the eigenvalues corresponding to $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then $A = PDP^{-1}$.

Proof. $\Rightarrow$. Given A is diagonalizable. So $A =$ ______________ where _______ is invertible and _______ is _____________________. So $AP =$ ______________. Let $P = [\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n]$. Then

$$AP = A[\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n] = [\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad]$$

while, since D is diagonal with diagonal entries $\lambda_1, \ldots, \lambda_n$, then

$$PD = [\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad]$$

But $AP = PD$ so $A\mathbf{v}_1 =$ ______________, so _______ is an ____________________________ corresponding to _______. Similarly for $\mathbf{v}_2 \ldots \mathbf{v}_n$. Since $P = [\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n]$ is invertible, $\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n$ are ____________________________by the Connections Theorem (IMT).

$\Leftarrow$ Given A has n independent eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ corresponding to $\lambda_1, \ldots, \lambda_n$ (λ's are not necessarily, distinct by the way). Let $P = [\mathbf{v}_1\, \mathbf{v}_2\, \ldots\, \mathbf{v}_n]$. P is ____________________________by the Connections Theorem (IMT). Let

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Then

$$AP = [\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad] = [\lambda_1\mathbf{v}_1\, \ldots\, \lambda_n\mathbf{v}_n] = PD$$

because $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are ___________________________________. So

$$AP = PD \Rightarrow A = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

So A is ___________________________________.

with (*LinearAlgebra*) :
Be careful when determining whether a stochastic matrix P is **regular**. Looking at P^2 may not be sufficient. higher powers of P may be required before all entries are positive. Here's a 4 x 4 matrix whose first three powers contain 0s, but whose fourth power has all positive entries. Suppose this were a mouse maze, could you interpret the transition matrix?

$$P := \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} & 0 \end{bmatrix} :$$

$P^2; P^3; P^4$

$$\begin{bmatrix} \frac{1}{4} & 0 & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{5}{16} & 0 \\ 0 & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{2} & 0 & \frac{1}{16} & \frac{5}{16} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{3}{32} \\ \frac{1}{4} & 0 & \frac{13}{64} & \frac{9}{64} \\ \frac{1}{8} & \frac{1}{16} & \frac{3}{64} & \frac{3}{32} \\ \frac{1}{4} & \frac{5}{16} & \frac{11}{32} & \frac{1}{64} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{8} & \frac{3}{32} & \frac{19}{128} & \frac{3}{32} \\ \frac{1}{8} & \frac{9}{64} & \frac{27}{128} & \frac{13}{256} \\ \frac{3}{32} & \frac{3}{32} & \frac{25}{256} & \frac{7}{256} \\ \frac{9}{32} & \frac{1}{64} & \frac{55}{256} & \frac{21}{128} \end{bmatrix} \quad \textbf{(1)}$$