Days 40 and 41: Intro to Stage-Matrix Models
Example 1: A population model with two classes: non-reproductive juveniles and reproductive adults.
$with(LinearAlgebra)$:

NOTE: The *interface(displayprecision)* command controls the number of decimal places to be displayed. Let's set it
to display only 4 digits (though internally it is keeping track of more digits)

$interface(displayprecision = 4)$

$$-1 \tag{1}$$

$$A := \begin{bmatrix} 0 & 5 \\ \frac{1}{8} & \frac{1}{2} \end{bmatrix}:$$

Determine both the eigenvalues and eigenvectors.
lambda, $v := evalf(Eigenvectors(A))$

$$\begin{bmatrix} 1.0792 \\ -0.5792 \end{bmatrix}, \begin{bmatrix} 4.6332 & -8.6332 \\ 1.0000 & 1.0000 \end{bmatrix} \tag{2}$$

We see that 1.079156198 is the **dominant** (largest magnitude) **eigenvalue**. Extract the corresponding eigenvector
and call it x
$x := v[1\,..,\,1]$

$$\begin{bmatrix} 4.6332 \\ 1.0000 \end{bmatrix}$$

(4)

Make it a probability vector. The command VectorNorm(x,1) adds the absolute value of the vectors components.

$$\frac{1}{VectorNorm(x,\,1)}x$$

$$\begin{bmatrix} 0.8225 \\ 0.1775 \end{bmatrix} \tag{5}$$

Other ways to get this same information if you have trouble using the commands above.

To get the eigenvalues and eigenvectors in decimal form:

$evalf(Eigenvectors(A))$

$$\begin{bmatrix} 1.0792 \\ -0.5792 \end{bmatrix}, \begin{bmatrix} 4.6332 & -8.6332 \\ 1.0000 & 1.0000 \end{bmatrix} \quad \textbf{(6)}$$

We see that 1.079156198 is the dominant (largest magnitude) eigenvalue. To make the corresponding eigenvector into
a vector with components that sum to 1, do the same thing you did when working with Markov chains: copy and paste
the components into the vector.

$$x\_new := \frac{1}{4.6332 + 1} \begin{bmatrix} 4.6332 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.8225 \\ 0.1775 \end{bmatrix} \quad \textbf{(7)}$$

Take-home Message: In the long run, 82.2% of the population will be juveniles and 17.78% will be adults.

Case Study: Northern Spotted Owl Extinction

Note: Choose different names for the matrices, eigenvalues, and eigenvectors so that you do not inadvertently use data from another example.

$$NSO := \begin{bmatrix} 0 & 0 & \frac{33}{100} \\ \frac{18}{100} & 0 & 0 \\ 0 & \frac{71}{100} & \frac{94}{100} \end{bmatrix}:$$

Determine both the eigenvalues and eigenvectors.

$lamNSO, vNSO := evalf(Eigenvectors(NSO))$

$$\begin{bmatrix} 0.9836 \\ -0.0218 + 0.2059\,\mathrm{I} \\ -0.0218 - 0.2059\,\mathrm{I} \end{bmatrix}, \begin{bmatrix} 0.3355 & -0.1678 - 1.5848\,\mathrm{I} & -0.1678 + 1.5848\,\mathrm{I} \\ 0.0614 & -1.3546 + 0.2900\,\mathrm{I} & -1.3546 - 0.2900\,\mathrm{I} \\ 1.0000 & 1.0000 & 1.0000 \end{bmatrix} \quad \textbf{(8)}$$

(9)

Notice that two of the eigenvalues are complex numbers. To find the dominant eigenvalue, we need to find the magnitude of each eigenvalue. In the handout I did it with a loop, but Spencer noticed that we can do it more simply:

abs(*lamNSO*)

$$\begin{bmatrix} 0.9836 \\ 0.2071 \\ 0.2071 \end{bmatrix} \quad (10)$$

Notice that the magnitude of each eigenvalue is less than 1. That means that powers of the eigenvalues get smaller and go to 0. Here we do need a loop to do the calculation:

for i **from** 1 **by** 1 **to** 3 **do** $\lim_{k \to infinity} (\text{abs}(\text{lamNSO}[i]))^k$ **end do**;

$$\begin{array}{c} 0.0000 \\ 0.0000 \\ 0.0000 \end{array} \quad (11)$$

As noted in the handout (see page 3: *Interpretation*), this means that the $A^k\mathbf{x}$ goes to $\mathbf{0}$ as k gets larger. In other words, the population becomes extinct.

Key point: The population will become extinct no matter what the starting population is. The eigenvalues depend only on the matrix A, not on the initial population.

Case Study: Northern Spotted Owl Growth. See **Problem 4 on Page 4 in the Handout**

Suppose that we were able to intervene and modify the survival rate of juvenile owls (say by protecting habitat or removing DDT from the environment).
If the juvenile survival rate improved to 50%, we would now have

$$G := \begin{bmatrix} 0 & 0 & \frac{33}{100} \\ \frac{50}{100} & 0 & 0 \\ 0 & \frac{71}{100} & \frac{94}{100} \end{bmatrix}:$$

Determine both the eigenvalues and eigenvectors.
$lamG, vG := evalf(Eigenvectors(G))$

$$\begin{bmatrix} 1.0469 \\ -0.0534 + 0.3302\,\mathrm{I} \\ -0.0534 - 0.3302\,\mathrm{I} \end{bmatrix}, \begin{bmatrix} 0.3152 & -0.1576 - 0.9738\,\mathrm{I} & -0.1576 + 0.9738\,\mathrm{I} \\ 0.1506 & -1.3992 + 0.4651\,\mathrm{I} & -1.3992 - 0.4651\,\mathrm{I} \\ 1.0000 & 1.0000 & 1.0000 \end{bmatrix} \quad \textbf{(12)}$$

Again two eigenvalues are complex and one is real. To determine the dominant eigenvalue,
abs(*lamG*)

$$\begin{bmatrix} 1.0469 \\ 0.3345 \\ 0.3345 \end{bmatrix} \quad \textbf{(13)}$$

This loop shows that powers of the two complex eigenvalues go to 0, but since the real eigenvalue is greater than 1, then its powers increase to infinity. The population will grow.

for i **from** 1 **by** 1 **to** 3 **do** $\lim_{k \to infinity} (\mathrm{abs}(\mathrm{lamG}[i]))^k$ **end do**;

$$Float(\infty)$$
$$0.0000$$
$$0.0000 \quad \textbf{(14)}$$

When the dominant eigenvalue is real and greater than 1, then we may determine the long-run distribution of the population among the age-classes by normalizing the eigenvector corresponding to this eigenvalue. First we get the eigenvector from above:
$xG := vG[1 .., 1]$

$$\begin{bmatrix} 0.3152 \\ 0.1506 \\ 1.0000 \end{bmatrix} \quad \textbf{(15)}$$

Then we normalize it
$\frac{1}{VectorNorm(xG, 1)} xG$

$$\begin{bmatrix} 0.2151 \\ 0.1027 \\ 0.6822 \end{bmatrix} \quad \textbf{(16)}$$

We interpret the result to mean that 21.5% of the long-run population will be juveniles, about 10.3% will be subadults, and 68.2% adults.
Key Point: This growth occurs no matter what the initial population is.