Stage-Matrix Models: Key Points

1. The equation $\mathbf{x}_k = A\mathbf{x}_{k-1} = A^k\mathbf{x}_0$ is a stage-matrix model of the population. A is an $n \times n$ matrix, where the population has been divided into n classes or stages.
2. The eigenvalues of A are calculated and listed in descending order of magnitude: $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$. This may be done using the `Maple` command:
$$abs(Eigenvalues(A)).$$
If $|\lambda_1|$ is strictly greater than $|\lambda_2|$, we call λ_1 the **dominant eigenvalue**.
3. If $|\lambda_1| < 1$, then the population will decrease to extinction, no matter what the initial population vector $\mathbf{x}_0$ is.
4. If λ_1 is a real number greater than 1 and all the other eigenvalues are less than 1 in magnitude, then the population is increasing exponentially (no matter what the initial population). In this case if $\mathbf{v}_1$ is the eigenvector corresponding to λ_1, then the normalized eigenvector $\frac{1}{||\mathbf{v}_1||_1}\mathbf{v}_1$ gives the percentages found in each class in the long-run population distribution. This vector can be calculated with the `Maple` command
$$\frac{1}{VectorNorm(\mathbf{v}_1, 1)}\mathbf{v}_1 \quad \text{or} \quad \frac{1}{|x_1| + |x_2| + \cdots + |x_n|}\mathbf{v}_1.$$

Here we are assuming $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

Sustainable Harvesting

If a population is increasing, one might be interested in harvesting a portion of the population for some purpose. An issue of major interest in this case is how much of the population might be harvested while maintaining the population at a constant level. This is surely an important issue in the forestry and fishing industries. Classes are not distinguished in the harvest, although this could be added to the model with only slightly more difficulty. If a fraction h of the population is harvested each year, where $0 \leq h < 1$, the population model becomes
$$\mathbf{x}_{k+1} = A\mathbf{x}_k - hA\mathbf{x}_k.$$
It is desired to find h so that the population at year $k+1$ equals that of year k. Letting $\mathbf{x}$ be this common population vector, then an h is sought with
$$\mathbf{x} = A\mathbf{x} - hA\mathbf{x} = (1-h)A\mathbf{x}.$$
Since this equation may be rewritten as
$$A\mathbf{x} = \frac{1}{1-h}\mathbf{x},$$
the number $\frac{1}{1-h}$ must be an eigenvalue of A. Since $h < 1$, then $\frac{1}{1-h} > 1$. If λ_1 is the only eigenvalue of A with magnitude larger than 1, it follows that we have
$$\frac{1}{1-h} = \lambda_1 \qquad \text{or} \qquad h = \frac{\lambda_1 - 1}{\lambda_1}.$$
In the increasing owl population above, $\lambda_1 = 1.0469$. Even though it is difficult to imagine why owls would be harvested, the appropriate calculation shows that
$$h = \frac{\lambda_1 - 1}{\lambda_1} = \frac{.0469}{1.0469} \approx 0.0447.$$
Thus, 4.47% of the population could be harvested each year and the population would remain constant.

`Maple` *Exercises: Due Monday*

Work with your partner.

☞ The interface(displayprecision) command controls the number of decimal places to be displayed. Let's set it to display only 4 digits (though internally it is keeping track of more digits). Use

interface(displayprecision = 4):

1. These data come from Levi Arthur's (H '05) final project for Math 214 and pertain to the white-tailed deer, *Odocoileus virginianus*, population in New York. It has three age-classe: fawns, subadults, and does (adult females).

> Each adult female gives birth on average to 0.52 female fawns per year
> 55% of female fawns survive to become subadults
> 60% of female subadult deer survive to become adults
> 90% of female adult deer survive to become adults

(*a*) Determine the stage-matrix for this problem.

(*b*) Show that the deer population is increasing.

(*c*) Determine the percentage of each class in the long-run population.

(*d*) Assume you work for the NY State Department of Environmental Conservation. Determine the sustainable harvest rate (read the other side) for white-tailed deer in New York. (This 'harvesting' rate applies to all age classes.)

☞ Rember to use fractions in `Maple` and then use *evalf( )* to convert answers to decimals.

2. This modeling technique may also be applied to plants. Instead of age classes, classes based on the size of plant are used. Instead of fecundity, the growth of the plant is called sprout production. In Reference 1, a population of a common shrub called the speckled alder, *Alnus incana ssp. rugosa*, was grouped into five size classes based on stem diameter: less than .1 cm, .1–.9 cm, 1–1.9 cm, 2–2.9 cm, and 3–3.9 cm. The number of stems with diameter of more than 4 cm was too small to allow meaningful measurement. The following matrix was derived for this situation:

$$\begin{bmatrix} .78 & .02 & .06 & .10 & .14 \\ .12 & .76 & 0 & 0 & 0 \\ 0 & .12 & .86 & 0 & 0 \\ 0 & 0 & .14 & .58 & 0 \\ 0 & 0 & 0 & .38 & .83 \end{bmatrix}.$$

(*a*) Interpret biologically what each entry in the first row of this matrix means.

(*b*) Interpret biologically what .58 in row 4, column 4 means.

(*c*) Determine whether the alder shoot 'population' is becoming 'extinct' in this model. If the population is not becoming extinct, determine the percentage of each class in the long-run population.

3. (*a*) **Extra Credit for Homework Grade** (Due Tuesday at 5:00 pm). Return to Problem 1 above. Suppose that you work for the NY State Department of Environmental Conservation. By issuing hunting permits for does (adult females) your department can lower the survival rate of adult females below the current 90%. What survival rate r would you need to achieve to produce a situation where the population is stable, that is the dominant eigenvalue is $\lambda_1 = 1$? You can solve this using trial and error by trying various values for r in the stage-matrix for deer and finding the eigenvalues. Your answer for r should give an eigenvalue correct to three decimal places, that is, λ_1 should be between 0.9995 and 1.0005.

(*b*) What would be the long-run population distribution in this case?

(*c*) Further Bonus: Determine r using a method other than trial and error. It is easy if you think about it in the correct way.