Practice Problems Math 204

#1 a) Strategy — make this a columnspace problem. Let

$$A = \begin{bmatrix} 1 & -3 & 8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 8 & -3 \\ 0 & -2 & 22 & -6 \\ 0 & 1 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 8 & -3 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 12 & 2 \end{bmatrix} \quad \text{3 pivots}$$

So a basis for Col A and H $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix} \right\}$

b) $\dim H = 3$

#2 Strategy form a matrix A & either row reduce to I or take det A

a)

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 8 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -1 \end{vmatrix} = 1(-1+2) = 1 \neq 0$$ Since $\det A \neq 0$,

A is invertible $\therefore$ cols of A span $\mathbb{R}^3$ and are independent (IMT) So B was a basis.

b) to find $[x]_B$, solve $A\vec{z} = \vec{x}$ ... i.e reduce $[A\ \vec{x}]$

$$[A\ \vec{x}] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 8 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 2 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad \begin{aligned} x_1 &= -2 \\ x_2 &= 0 \\ x_3 &= 5 \end{aligned} \qquad [x]_B = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} \checkmark$$

#3 Use the standard basis $B = \{1, t, t^2\}$ for $\mathbb{P}_2$.

a) $[p_1]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $[p_2]_B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $[p_3]_B = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$. Now form a matrix A and check $|A|$ (see #2)

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -5 \end{vmatrix} = 1(5-2) \neq 0$$ so A is invertible

which implies the cols of A are independent & span $\mathbb{R}^3$, so are a basis for $\mathbb{R}^3$. Consequently $B = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ are a basis for $\mathbb{P}_2$.

b) Careful ... we are given $[q]_B = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$ ... This means that

$$\vec{q} = -3\vec{p}_1 + \vec{p}_2 + 2\vec{p}_3 = -3(1+t^2) + (2-t+3t^2) + 2(1+2t-4t^2)$$
$$= 1 + 3t - 8t^2$$

#4 Strategy reduce A to find pivot cols ... and also Nul A

$$\begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & -2 & -5 & -3 \\ 0 & 2 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 1 & 5/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & 6 & 5 & 0 \\ 0 & 1 & 5/2 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

pivots ... cols 1 & 2

For Nul A

Basis for Col A $= \left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}$

Nul A:

$x_1 = -6x_3 - 5x_4$

$x_2 = -5/2\, x_3 - 3/2\, x_4$

x_3, x_4 free

$$x_3 \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$$

Basis for Nul A: $\left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

dim Col A = 2 ; dim Nul A = 2

#5 Use coordinates and form a matrix A -- find its pivots & convert back to M_{22}

pivots

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 8 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 2 & -2 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 8 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 8 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So a basis for Span S is: $\left\{ \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 5 \\ 2 & 8 \end{bmatrix} \right\}$ dim = 3

#6 As discussed in class (Day 33), If $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis for $\mathbb{R}^n$, then the change of coordinates matrix from the $\mathcal{B}$ basis to the standard basis is $P_{\mathcal{B}} = [\vec{b}_1 \dots \vec{b}_n]$

But we want to reverse this... to go from the standard basis to the $\mathcal{B}$ basis... so we must use $P_{\mathcal{B}}^{-1}$

Here $P_{\mathcal{B}} = \begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix}$, det $P_{\mathcal{B}} = 1$ so $P_{\mathcal{B}}^{-1} = \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}$

#7 $T: M_{22} \to \mathbb{R}^2$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+2d \\ b+c-d \end{bmatrix}$. Find ker T

$A \in \ker T \iff T(A) = \vec{0}_2 \iff T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+2d \\ b+c-d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so

Solution: $a = -2d$ 2 free variables, c & d
$b = -c+d$

just like parametric solutions

$\ker T = \left\{ \begin{bmatrix} -2d & -c+d \\ c & d \end{bmatrix} ; c, d \in \mathbb{R} \right\}$ so a basis $\left\{ \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$

$\dim \ker T = 2$

#8 $T: \mathbb{R}^2 \to \mathbb{P}_2$ by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (a+b) + at + (b-a)t^2$

a) Linear?
Compare $T(\vec{u}+\vec{v})$ to $T(\vec{u}) + T(\vec{v})$ where $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix}$

$$\begin{aligned} T(\vec{u}+\vec{v}) = T\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right) &= ((a+c)+(b+d)) + (a+c)t + ((b+d)-(a+c))t^2 \\ &= (a+b)+(c+d) + at + ct + (b-a)t^2 + (d-c)t^2 \\ &= (a+b) + at + (b-a)t^2 \\ &\quad + (c+d) + ct + (d-c)t^2 \\ &= T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) \checkmark \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

b) Show T is one-to-one
<u>Proof</u> T is one-to-one $\iff \ker T = \{\vec{0}_2\}$

So determine ker T:

$$\begin{bmatrix} a \\ b \end{bmatrix} \in \ker T \iff T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (a+b) + at + (b-a)t^2 = \underbrace{0 + 0t + 0t^2}_{\vec{0}(t)}$$

compare coefficients

$$\iff \left.\begin{aligned} a+b &= 0 \\ a &= 0 \\ b-a &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 0+b &= 0 \\ & \Rightarrow b = 0 \\ b-0 &= 0 \end{aligned}\right\} \text{ So } a = b = 0$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \in \ker T \iff \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}_2 \quad \therefore \ker T = \{\vec{0}\}$$

$\therefore$ T is one-to-one

More Practice

#9) Is $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b^2 \end{bmatrix}$ a linear transf?

a) compare $T(\vec{u}+\vec{v})$ to $T(\vec{u})+T(\vec{v})$ where $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix}$

$$T(\vec{u}+\vec{v}) = T\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right) = \begin{bmatrix} (a+c)+(b+d) \\ (b+d)^2 \end{bmatrix}$$

not equal

$$T(\vec{u})+T(\vec{v}) = T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b^2 \end{bmatrix} + \begin{bmatrix} c+d \\ d^2 \end{bmatrix} = \begin{bmatrix} a+b+c+d \\ b^2+d^2 \end{bmatrix}$$

Not a linear transformation

b)

$T: \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\vec{p}) = \begin{bmatrix} p(2) \\ 2p(0) \end{bmatrix}$ ... Linear?

Check Let $\vec{p}, \vec{q} \in \mathbb{P}_2$

$$T(\vec{p}+\vec{q}) = \begin{bmatrix} (p+q)(2) \\ 2(p+q)(0) \end{bmatrix} = \begin{bmatrix} p(2)+q(2) \\ 2(p(0)+q(0)) \end{bmatrix} = \begin{bmatrix} p(2) \\ 2p(0) \end{bmatrix} + \begin{bmatrix} q(2) \\ 2q(0) \end{bmatrix} = T(\vec{p})+T(\vec{q}) \checkmark$$

For any scalar c

$$T(c\vec{p}) = \begin{bmatrix} (cp)(2) \\ 2(cp)(0) \end{bmatrix} = \begin{bmatrix} c\,p(2) \\ 2c\,p(0) \end{bmatrix} = c\begin{bmatrix} p(2) \\ 2p(0) \end{bmatrix} = c\,T(\vec{p}) \checkmark$$

$\therefore$ linear

If $\vec{p} = a+bt+ct^2$

$$\vec{p} \in \ker T \iff T(a+bt+ct^2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} p(2) \\ 2p(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \overset{p(2)}{a+2b+4c} \\ \underset{2p(0)}{2a} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff \left.\begin{matrix} a+2b+4c=0 \\ 2a=0 \end{matrix}\right\} \Rightarrow \begin{matrix} a=0 \\ b+2c=0 \end{matrix} \Rightarrow \begin{matrix} b=-2c \\ a=0 \end{matrix}$$

$\ker T = \{-2ct+ct^2 : c \in \mathbb{R}\}$. Basis is $\{-2t+t^2\}$

#10 Is $W = \left\{\begin{bmatrix} a \\ a^2 \end{bmatrix} : a \in \mathbb{R}\right\}$ a subspace of $\mathbb{R}^2$?

a) Is $\vec{0} \in W$ ... yes let $a=0$, then $\begin{bmatrix} a \\ a^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W$

b) closed under addition? Let $\begin{bmatrix} a \\ a^2 \end{bmatrix}, \begin{bmatrix} b \\ b^2 \end{bmatrix} \in W$. Is $\begin{bmatrix} a \\ a^2 \end{bmatrix} + \begin{bmatrix} b \\ b^2 \end{bmatrix}$ in W?

$$\begin{bmatrix} a \\ a^2 \end{bmatrix} + \begin{bmatrix} b \\ b^2 \end{bmatrix} = \begin{bmatrix} a+b \\ a^2+b^2 \end{bmatrix} \neq \begin{bmatrix} a+b \\ (a+b)^2 \end{bmatrix}$$

so Not closed under addition

$\therefore$ Not a subspace of $\mathbb{R}^2$

#11 is $W = \{p \in \mathbb{P}_5 : \int_0^1 p(t)\,dt = 1\}$ a subspace of $\mathbb{P}_5$

No, $\vec{0}(t) \notin W$ because $\int_0^1 \vec{0}(t)\,dt = \int_0^1 0\,dt = 0\int_0^1 dt = 0 \neq 1$

#12 Let $T: V \to W$ be linear. Show ker T is a subspace of $\vec{V}$

(a) Show $\vec{0}_V \in \ker T$. Well $T(\vec{0}_V) = T(0\cdot\vec{0}_V) = 0T(0\vec{v}) = \vec{0}_W$

So $\vec{0}_V \in \ker T$

(b) Let $\vec{u}, \vec{v} \in \ker T$. Show $\vec{u}+\vec{v} \in \ker T$

$$T(\vec{u}+\vec{v}) \overset{\text{linear}}{=} T(\vec{u}) + T(\vec{v}) \overset{\vec{u},\vec{v} \in \ker T}{=} \vec{0}+\vec{0} = \vec{0} \quad \therefore \vec{u}+\vec{v} \in \ker T$$

(c) Let c be any scalar and $\vec{u} \in \ker T$. Show $c\vec{u} \in \ker T$

But

$$T(c\vec{u}) \overset{\text{linear}}{=} cT(\vec{u}) \overset{\vec{u} \in \ker T}{=} c\cdot\vec{0} = \vec{0} \quad \therefore c\vec{u} \in \ker T$$

$\therefore$ ker T is a subspace of $\vec{V}$

#13 p 223 #21 $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x^2+y^2 \le 1 \right\}$

(a) $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in H$ since $0^2+0^2 \le 1$

(b) Not closed under addition. Eg $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{u}, \vec{v} \in H$ since $1^2+0^2 \le 1$ and $0^2+1^2 \le 1$

But $\vec{u}+\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin H$ since $1^2+1^2 \not\le 1^2$

So Not a subspace of $\mathbb{R}^2$

#14 Return to p 255 #32 Find $[\vec{p}]_B$ if $\vec{p} = 3+t+5t^2$

Use coordinates & Row Reduction. Solve $\left[[p_1]_S \; [p_2]_S \; [p_3]_S \; [p]_S \right]$ where S is the standard basis $\{1, t, t^2\}$ for $\mathbb{P}_2$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 1 & 3 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad \text{so } [\vec{p}]_B = \begin{bmatrix} 10 \\ -3 \\ -1 \end{bmatrix}$$

Check $10\vec{p}_1 - 3\vec{p}_2 - \vec{p}_3 = 10(1+t^2) - 3(2-t+3t^2) - (1+2t-4t^2)$
$= 3+t+5t^2$ ✓

#15 Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear and one-to-one
Let A be the standard matrix for T, A is $m \times n$
Remember

$$\text{Range } T = \text{Col } A \quad \text{and} \quad \text{Ker } T = \text{Nul } A$$

Since T is one-to-one $\ker T = \{\vec{0}_n\} = \text{Nul } A$
So $A\vec{x} = \vec{0}$ has only the trivial solution.
This means the columns of A are independent
So each column has a pivot ... but the pivot cols of A form a basis for Col A. Since there are n columns, there are n pivots and n vectors in the basis

$$\text{so} \quad \dim \text{Col } A = n = \dim \text{Range } T$$

#16 $H = \text{Span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4\}$ and $\dim H = 2$

So two vectors are a basis for H. Pick any 2 vectors in the set so that neither is a scalar multiple of the other. Then they are independent and must span H because $\dim H = 2$ (Thm 12)

#17 p 270 #21 Let $A\vec{x} = \vec{b}$ represent the general system, where A is 9×10. Since the system is consistent for all $\vec{b} \in \mathbb{R}^9$, by Thm 1.4 there's a pivot in every row of $A \Rightarrow A$ has 9 pivots $\Rightarrow$ Rank $A = 9$. Since

$$\text{Rank } A + \dim \text{Nul } A = 10$$

then $\dim \text{Nul } A = 1$.

This means a single vector $\vec{v}$ is a basis for Nul A
i.e. $\text{Nul } A = \text{span}\{\vec{v}\}$
So every solution of the homog sys $A\vec{x} = \vec{0}$ is a scalar multiple of $\vec{v}$.

It is impossible to find two solutions that are <u>not</u> multiples of each other