1) a) Is $\vec{0}(t) \in W$? $\vec{0}(1)=0$ and $\vec{0}(2)=0$ ∴ so $\vec{0}(1) = 4\vec{0}(2)$ ✓

Let $f, g \in W$ is $f+g \in W$. check: $f, g \in W \Rightarrow f(1) = 4f(2)$ and $g(1) = 4g(2)$ so

$$(f+g)(1) = f(1)+g(1) = 4f(2)+4g(2) = 4(f(2)+g(2))$$
$$= 4(f+g)(2) \checkmark$$

Let $f \in W$. Is $cf \in W$: Check: $f \in W$ so $f(1) = 4f(2)$. So

$(cf)(1) = c(f(1)) = c(4f(2)) = 4(cf(2)) = 4(cf)(2)$ ✓

∴ Subspace

b) Is $\vec{0} \in W$? $\vec{0}(3)+\vec{0}(2) = 0+0 = 0 \neq 1$ ∴ $\vec{0} \notin W$

∴ not a subspace

2) $T(\vec{p}+\vec{q}) = t\underbrace{(\vec{p}+\vec{q})'}_{\text{derivative of a sum = sum of derivatives}}(t) = t(\vec{p}\,'+\vec{q}\,')(t) = t\vec{p}\,'(t)+t\vec{q}\,'(t)$

$= T(\vec{p}) + T(\vec{q})$ ✓

pull constant out

$T(c\vec{p}) = t(c\vec{p})'(t) = t(c\,\vec{p}\,'(t))$

$= c(t\vec{p}\,'(t)) = cT(\vec{p})$ ✓

#3 a) convert using coordinates... form a matrix column

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

. The cols of A are basis ⟺ $|A| \neq 0$ (IMT).

$$|A| = 1\begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1-1 = -2 \neq 0 \ldots$$

∴ S is a basis of $\mathbb{P}_2$

b) Put the matrices in as columns

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

either take det B or row Reduce

$$B \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim I_4$$

∴ The matrices are a basis for $M_{2\times 2}$

c) No. $\dim M_{2\times2} = 4$. There are 5 'vectors'; so this not a basis

d) see Next Page

4 a) A and B similar $\Rightarrow A = PBP^{-1} \Rightarrow |A| = |PBP^{-1}|$

$$= \cancel{|P|}\,|B|\,\cancel{|P^{-1}|}$$
$$= |B|$$

So $|A|$ and $|B|$ have the same determinant. If A is not invertible $\Rightarrow |A| = 0 \Rightarrow |B| = 0 \Rightarrow$ B not invertible

b) A similar to B $\Rightarrow A = PBP^{-1}$
B " C $\Rightarrow B = QCQ^{-1}$) P need not be Q !

$\therefore A = PBP^{-1} = P(QCQ^{-1})P^{-1} = (PQ)C(Q^{-1}P^{-1}) = (PQ)C(PQ)^{-1}$

If we let $R = PQ$, then $A = RCR^{-1}$ $\therefore$ A and C are similar

$\Rightarrow$ 3 d) Use coordinates again --- vectors go in as columns. Solve

$$\begin{array}{cccc} p_1 & p_2 & p_3 & q \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & -1 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \begin{array}{l} x_1 = 3 \\ x_2 = -1 \\ x_3 = -2 \end{array}$$

$\vec{q} = 3\vec{p_1} - \vec{p_2} - 2\vec{p_3}$

#5 B is a basis $\Leftrightarrow$ $\begin{vmatrix} 1 & 2 & 2 \\ 2 & k & 4 \\ 0 & k & k \end{vmatrix} \neq 0 \Leftrightarrow 1\begin{vmatrix} k & 4 \\ k & k \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ k & k \end{vmatrix} = k^2 - 4k = 0$

$k(k-4) = 0$

$k = 0, 4$

#6a) No. Too few vectors. A basis of $\mathbb{R}^3$ has 3 vectors

b) Need $\begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 5 \\ 4 & 2 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & 5 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} = -10 - 4 = -14 \neq 0$

$\therefore$ a basis by IMT

c) No; too many vectors (4); $\dim \mathbb{R}^3 = 3$

d) No. The first and last rows are multiples of each other so the determinant is 0

#7 $W = \left\{ a\underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{u} + b\underbrace{\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}}_{v} + c\underbrace{\begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}}_{w} : a, b, c \in \mathbb{R} \right\} = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$

$\therefore$ W is automatically a subspace of $\mathbb{R}^3$ (Thm 1, p 221)

#8 a) $P = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}$

b) $x_1 = Px_0 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .85 \\ .15 \end{bmatrix}$ $\quad x_2 = Px_1 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .85 \\ .15 \end{bmatrix} = \begin{bmatrix} .875 \\ .125 \end{bmatrix}$

c) $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $x_1 = Px_0 = \begin{bmatrix} .95 \\ .05 \end{bmatrix}$ $\quad x_2 = Px_1 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .95 \\ .05 \end{bmatrix} = \begin{bmatrix} .925 \\ .075 \end{bmatrix}$

$\vec{q}$ = eigenvector for $\lambda = 1$

$[P-I \; 0] = \begin{bmatrix} -.05 & .45 & 0 \\ .05 & -.45 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ $\quad \vec{q} = \frac{1}{10}\vec{x} = \begin{bmatrix} .9 \\ .1 \end{bmatrix}$

Practice Math 204

#9 See back of text

#10 ① Eigenvalues

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)\left[\lambda^2-3\lambda+2\right]$$

$$= (2-\lambda)(\lambda-1)(\lambda-2) = 0$$

$\lambda = 2$ (Alg mult 2), $\lambda = 1$

② Eigenvectors

$\lambda = 2$

$$[(A-2I)\ 0] = \begin{bmatrix} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad x = \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$\lambda = 1$

$$[(A-I)\ 0] = \begin{bmatrix} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$$

$$P = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$