Coverage The exam covers Sections 1.8 and 1.9 on Linear Transformations. Sections 2.1, 2.2, and 2.3 on Matrix Algebra and finding Matrix Inverses. Sections 3.1 and 3.2 on Determinants.

Linear Transformations

DEFINITION 0.0.1. A transformation T is **linear** if both

(*a*) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T.

(*b*) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

THEOREM 0.0.2. If T is a linear transformation, then

(*a*) $T(\mathbf{0}) = \mathbf{0}$

(*b*) $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{d})$

(*c*) and more generally $T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(v_1) + \cdots + c_pT(v_p)$.

THEOREM 0.0.3 (Theorem 1.10). Let $T : \mathbb{R}^n \to \mathbb{R}^m$ b a linear transformation. Then there is a unique matrix A called the **standard matrix for** T such that $T(x) = Ax$. In fact, A is the matrix

$$A = [T(\mathbf{e}_1)\, T(\mathbf{e}_2)\, \cdots\, T(\mathbf{e}_n)].$$

DEFINITION 0.0.4. A mapping $T : X \to Y$ is **onto** if for every $\mathbf{b} \in Y$, there is an $\mathbf{x} \in X$ so that $T(\mathbf{x}) = \mathbf{b}$. (Alternatively: Each $\mathbf{b}$ in Y is the image of at least one $\mathbf{x}$ in X.)

A mapping $T : X \to Y$ is **one-to-one** if for whenever $\mathbf{u}$ and $\mathbf{v}$ are in the domain X and $\mathbf{u} \neq \mathbf{v}$, then $T(\mathbf{u} \neq T(\mathbf{v})$. (Alternatively: Each $\mathbf{b}$ in Y is the image of at most one $\mathbf{x}$ in X. Alternatively: If $\mathbf{u}$ and $\mathbf{v}$ are in the domain X and $T(\mathbf{u}) = T(\mathbf{v})$, then $\mathbf{u} = \mathbf{v}$.)

You can use $\mathbb{R}^n$ and $\mathbb{R}^m$ for X and Y. But later we will see that X and Y represent more general vector spaces.

1. Four questions about a transformation T:

(*a*) Is it linear? Check $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(c\mathbf{u}) = cT(\mathbf{u})$.

(*b*) If it is linear, what is its standard matrix?

(*c*) Is it onto?

(*d*) Is it one-to-one?

THEOREM 0.0.5 (Theorem 1.11). Let $T : \mathbb{R}^n \to \mathbb{R}^m$ b a linear transformation. Then T is one-to-one if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution, i.e., $\mathbf{x} = \mathbf{0}$.

THEOREM 0.0.6 (The Onto Dictionary). Let A be an $m \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. Then the following are equivalent.

(*e*) A has m pivots positions

(*f*) A has pivot in every row

(*g*) For any $\mathbf{b} \in \mathbb{R}^m$, the system $A\mathbf{x} = \mathbf{b}$ is consistent

(*h*) Any $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of A

(*i*) The columns of A span $\mathbb{R}^m$

(*j*) $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto

THEOREM 0.0.7 (The One-to-One Dictionary). Let A be an $m \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. Then the following are equivalent.

(*e*) A has n pivots positions

(*f*) A has pivot in every column (no free variables)

(*g*) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$

(*h*) The columns of A are independent

(*i*) $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one

Matrix Algebra

THEOREM 0.0.8 (See Theorem 2.1 in text)**.** Basic Matrix Algebra for addition and scalar multiplication of matrices.

DEFINITION 0.0.9. If A is $m \times n$ and B is $n \times p$ with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then

$$AB = A[\mathbf{b}_1 \; \mathbf{b}_2 \; \ldots \; \mathbf{b}_p] = [A\mathbf{b}_1 \; V\mathbf{b}_2 \; \ldots \; A\mathbf{b}_p]$$

THEOREM 0.0.10. Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B. (Remember $A\mathbf{b}_j$ is just the combination combination of the columns of A using the weights from $\mathbf{b}_j$.

THEOREM 0.0.11 (See Theorem 2.2 in text)**.** Basic Matrix Algebra for multiplication of matrices.

Warning. (1) In general $Ab \neq BA$.

(2) Cancellation fails. $AB = AC$ does not mean $B = C$.

(3) If $AB = 0$, it does not follow that $A = 0$ or $B = 0$. □

THEOREM 0.0.12 (Theorem 2.2)**.** Basic Matrix Algebra for transposes of matrices.

DEFINITION 0.0.13. An $n \times n$ matrix A is **invertible** or **non-singular** if there is an $n \times n$ matrix C so that $AC = I$ and $CA = I$.

FACT 0.1. Inverses are unique, so we may denote the inverse of A, if it exists, by A^{-1}. Moreover, by the Connections Theorem, if A and C are both $n \times n$ and $CA = I$, then $C = A^{-1}$. Similarly, if $AC = I$, then $C = A^{-1}$. That is, we only need to check multiplication by C on one side.

THEOREM 0.0.14 (Theorem 2.4)**.** How to find the inverse of a 2×2 matrix, when it exists.

THEOREM 0.0.15 (Theorem 2.5, First Connection)**.** If A is an invertible $n \times n$ matrix, then for each $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ has a *unique* solution. (This is now part of the larger Connections Theorem.)

THEOREM 0.0.16 (Theorem 2.6, includes Socks and Shoes)**.** The algebra of inverses. Remember to use Socks and Shoes you must know that both A and B are invertible before stating that $(AB)^{-1} = B^{-1}A^{-1}$.

DEFINITION 0.0.17. An $n \times n$ **elementary matrix** E is one that is obtained by performing a single row operation on I_n.

THEOREM 0.0.18 (Elementary Fact 1). If an elementary row operation is performed on an matrix A, the resulting matrix can be written as EA, where E is the matrix created by performing the same operation on I.

THEOREM 0.0.19 (Elementary Fact 2). Every elementary matrix is invertible.

You should be able to justify why an elementary matrix is invertible and be able to determine its inverse.

THEOREM 0.0.20 (Theorem 2.7). An $n \times n$ matrix A is invertible if and only if A is row equivalent to I. In this case $[A\ I] \sim [I\ A^{-1}]$ and so both A^{-1} and A can be written as a product of (invertible) elementary matrices.

2. You should be able to determine whether an $n \times n$ matrix A is invertible—and if so find its inverse—via row reduction.

THEOREM 0.0.21 (The Connections Theorem). Let A be an $n \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^n$ with standard matrix A. Then the following are equivalent.

(*a*) A is non-singular

(*b*) $A \sim I_n$

(*c*) A has n pivots positions

(*d*) A has pivot in every row

(*e*) For any $\mathbf{b} \in \mathbb{R}^n$, the system $A\mathbf{x} = \mathbf{b}$ is consistent

(*f*) Any $\mathbf{b} \in \mathbb{R}^n$ is a linear combination of the columns of A

(*g*) The columns of A span $\mathbb{R}^n$

(*h*) $T : \mathbb{R}^n \to \mathbb{R}^n$ is onto

(*i*) A has pivot in every column (no free variables)

(*j*) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$

(*k*) The columns of A are independent

(*l*) $T : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one

(*m*) There is an $n \times n$ matrix C such that $CA = I_n$

(*n*) There is an $n \times n$ matrix D such that $AD = I_n$

(*o*) A^T is invertible

(*p*) For any $\mathbf{b} \in \mathbb{R}^n$, the system $A\mathbf{x} = \mathbf{b}$ has a *unique* solution

3. How to use the Connections Theorem properly. On the test during a proof, one might say:

(*a*) Since the columns of the $n \times n$ matrix A are linearly independent, one of the conditions of the Connections Theorem is true, so they all are. Therefore the matrix A^T is invertible.

(*b*) Since A is $n \times n$ and $A\mathbf{x} = \mathbf{0}$ has more than one solution, one of the conditions of the Connections Theorem is false, so they all are. Therefore the corresponding transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is not onto.

Remember that the Connections Theorem only applies to $n \times n$ matrices, so you should mention that your matrix is square.

Determinants

Make sure you know the definition of determinant and cofactor.

DEFINITION 0.0.22. The **determinant** of a 1×1 matrix $A = [a]$ is $\det A = a$. For $n \ge 2$, the **determinant** of the $n \times n$ matrix $A = [a_{ij}]$ is the alternating sum

$$\det A = |A| = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots (-1)^{1+n} a_{1n} \det A_{1n}.$$

$C_{ij} = (-1)^{i+j} a_{ij} A_{ij}$ is the **(i,j)-cofactor** of A, then

$$\det A = |A| = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}.$$

THEOREM 0.0.23 (Cofactor Expansion)**.** If A is an $n \times n$ triangular matrix, then $\det A = a_{11} \cdot a_{22} \cdots a_{nn}$, that is, $\det A$ is the product of the diagonal entries of A.

THEOREM 0.0.24 (Triangular Determinants)**.** The determinant of an $n \times n$ matrix A can be computed by cofactor expansion along any row or any column.

THEOREM 0.0.25 (Theorem 3.3: Row Ops and Determinants)**.** Let A is an $n \times n$ matrix.

(*a*) If a multiple of one row of A is added to another row to produce a new matrix B, then $\det B = \det A$.

(*b*) If two rows of A are interchanged to produce B, then $\det B = -\det A$.

(*c*) If one row of A is multiplied by k to produce B, then $\det B = k \det A$.

THEOREM 0.0.26 (Another Connection: Determinants and Inverses)**.** An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

So we could add another condition to the Connections Theorem:

(*q*) $\det A \neq 0$