Linear Systems

THEOREM 1.0.1 (Theorem 1.1). Uniqueness of Reduced Row-Echelon Form

THEOREM 1.0.2 (Theorem 1.2). Existence and Uniqueness Theorem

THEOREM 1.0.3. Algebra of Scalar Multiplication and Vector Addition

THEOREM 1.0.4 (Theorem 1.3). Equivalent Representations Theorem

THEOREM 1.0.5 (Theorem 1.4). Spanning and Pivots Theorem: These are equivalent

THEOREM 1.0.6 (Theorem 1.5). Properties (Linearity) of Matrix-Vector Multiplication

THEOREM 1.0.7. Homogeneous Systems and Homogeneous Solutions Theorem

THEOREM 1.0.8. Independence of Matrix Columns

THEOREM 1.0.9. Characterization of Dependent Sets

THEOREM 1.0.10. Surplus of Vectors

THEOREM 1.0.11. The Zero Vector and Dependence

Linear Transformations

THEOREM 1.0.12. Properties of Linear Transformations

THEOREM 1.0.13 (Theorem 1.10). Standard Matrix Theorem

THEOREM 1.0.14. The Onto Dictionary

THEOREM 1.0.15. The One-to-One Dictionary

Matrix Algebra

THEOREM 2.0.1. Basic matrix algebra for addition and scalar multiplication.

THEOREM 2.0.2 (Theorem 2.2). Basic matrix algebra for multiplication of matrices.

THEOREM 2.0.3 (Theorem 2.3). Algebra of Transposes.

THEOREM 2.0.4 (Theorem 2.4). 2×2 Matrix Inverses.

THEOREM 2.0.5 (Theorem 2.5). First Connection.

THEOREM 2.0.6 (Theorem 2.6, includes Socks and Shoes). The algebra of inverses.

THEOREM 2.0.7. Elementary Matrix Facts 1, 2, and 3

THEOREM 2.0.8. The Connections Theorem

Determinants

THEOREM 3.0.1. Cofactor Expansion.

THEOREM 3.0.2. Triangular Determinants

THEOREM 3.0.3 (Theorem 3.3). Row Ops and Determinants

THEOREM 3.0.4. Elementary Matrix Determinants

THEOREM 3.0.5 (Theorem 3.4). Determinants and the Connections Theorem

New For Test 3

THEOREM 3.0.6 (Theorem 3.5). Determinants of Transposes

THEOREM 3.0.7 (Theorem 3.6). Determinants of Products

THEOREM 3.0.8. Determinants of Inverses

THEOREM 3.0.9. Determinants of Scalar Multiples kA

THEOREM 3.0.10 (Theorem 3.9). Determinants and Area and Volume

THEOREM 3.0.11 (Theorem 3.10). Transformed Area and Volume

Vector Spaces

DEFINITION 4.0.1. Definition of a Vector Space

THEOREM 4.0.2. Basic Properties of Vector Spaces. (See box on page 191.)

DEFINITION 4.0.3. Definition of a Subspace

THEOREM 4.0.4. Subspaces are Vector Spaces

THEOREM 4.0.5 (Theorem 4.1). Spans are Subspaces

DEFINITION 4.0.6. The **null space** of an $m \times n$ matrix A

THEOREM 4.0.7 (Theorem 4.2). Nul A is a Subspace

DEFINITION 4.0.8. The **column space** of an $m \times n$ matrix A

THEOREM 4.0.9 (Theorem 4.2). Col A is a subspace

DEFINITION 4.0.10. A **linear transformation** $T : \mathbb{V} \to \mathbb{W}$

DEFINITION 4.0.11. $T : \mathbb{V} \to \mathbb{W}$ is **onto**.

DEFINITION 4.0.12. $T : \mathbb{V} \to \mathbb{W}$ is **one-to-one**

THEOREM 4.0.13. The Three Kernel Facts Theorem

DEFINITION 4.0.14. **Linear independence** and **dependence** (same as before).

THEOREM 4.0.15 (Theorem 4.4). Characterization of Linear Dependence.

DEFINITION 4.0.16. **Basis** of a vector space.

Know the standard bases of $\mathbb{R}^n$, $\mathbb{P}_n$, $M_{m\times n}$.

THEOREM 4.0.17 (Theorem 4.5). Spanning Set Theorem

THEOREM 4.0.18 (Theorem 4.6). Basis of Col A.

THEOREM 4.0.19 (Theorem 4.7). Unique Representation Theorem.

DEFINITION 4.0.20. $\mathcal{B}$-**coordinates**, $[\mathbf{x}]_{\mathcal{B}}$.

DEFINITION 4.0.21. Isomorphism.

THEOREM 4.0.22 (Theorem 4.8). The Coordinate Mapping Theorem

THEOREM 4.0.23 (Theorem 4.9)**.** More Vectors than in a Basis Theorem

THEOREM 4.0.24 (Theorem 4.10)**.** Basis have the Same Size

DEFINITION 4.0.25. **Dimension of a Vector Space**

THEOREM 4.0.26 (Theorem 4.11)**.** Inflation or Expansion Theorem

THEOREM 4.0.27 (Theorem 4.12)**.** Basis Theorem

New Since Test 3

DEFINITION 4.0.28. **Row space of a matrix**

THEOREM 4.0.29 (Theorem 4.13)**.** Row equivalent matrices have the same row space.

THEOREM 4.0.30 (Theorem 4.14)**.** The Rank Theorem. (Remember we proved a better version than in the text that includes information about A^T.)

THEOREM 4.0.31 (Theorem 4.15)**.** Connections Extension to Rank, Row A, Nul A, and Col A.

DEFINITION 4.0.32. **Probability vector**

See Page 254.

DEFINITION 4.0.33. **Stochastic matrix**

DEFINITION 4.0.34. **Markov chain**

DEFINITION 4.0.35. **Steady-state vector**

DEFINITION 4.0.36. **Regular** stochastic matrix

THEOREM 4.0.37 (Theorem 4.18)**.** Regular stochastic matrices have a unique steady-state vector.

Eigenvalues and Eigenvectors

DEFINITION 5.0.1. **Eigenvector** and **eigenvalue**.

DEFINITION 5.0.2. **Eigenspace** of A corresponding to λ.

THEOREM 5.0.3 (From Classwork)**.** Three Eigenfacts

THEOREM 5.0.4. Eigenspaces are subspaces.

THEOREM 5.0.5 (Theorem 5.1)**.** Eigenvalues of triangular matrices.

THEOREM 5.0.6 (Theorem 5.2—Jack's Proof)**.** Eigenvectors corresponding to *distinct* eigenvalues

DEFINITION 5.0.7. **Characteristic equation** or **polynomial**.

DEFINITION 5.0.8. **Similar matrices**.

THEOREM 5.0.9 (Theorem 5.4)**.** Eigenvalues of similar matrices.

DEFINITION 5.0.10. **Diagonalizable matrix**.

THEOREM 5.0.11 (Theorem 5.5)**.** Diagonalization theorem.

THEOREM 5.0.12 (Theorem 5.6)**.** An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

THEOREM 5.0.13 (Theorem 5.7(b))**.** A is diagonalizable if and only if the sum of the dimensions of its eigenspaces is n.

THEOREM 5.0.14 (Stage-Matrix Models: Key Points)**.** 1. The equation $\mathbf{x}_k = A\mathbf{x}_{k-1} = A^k\mathbf{x}_0$ is a stage-matrix model of the population. A is an $n \times n$ matrix, where the population has been divided into n classes or stages.

2. The eigenvalues of A are calculated and listed in descending order of magnitude: $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$. If $|\lambda_1|$ is strictly greater than $|\lambda_2|$, we call λ_1 the **dominant eigenvalue**.
3. If $|\lambda_1| < 1$, then the population will decrease to extinction, no matter what the initial population vector $\mathbf{x}_0$ is.
4. If λ_1 is a real number greater than 1 and all the other eigenvalues are less than 1 in magnitude, then the population is increasing exponentially (no matter what the initial population). In this case if $\mathbf{v}_1$ is the eigenvector corresponding to λ_1, then the normalized eigenvector
$$\frac{1}{||\mathbf{v}_1||_1}\mathbf{v}_1 = \frac{1}{|x_1| + |x_2| + \cdots + |x_n|}\mathbf{v}_1$$
gives the percentages found in each class in the long-run population distribution.