Math 204: Test 3

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1. (8 pts) Assume A, B, and C are 4×4 matrices and that $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$. If $\det A = -3$, evaluate each of the following. (No explanation required.)

a) $\det A^{-1} =$

b) $\det(2A) =$

c) If $\det(BA) = -12, \text{then} \det B =$

d) $\det \begin{bmatrix} e & f & g-2e & h \\ a & b & c-2a & d \\ i & j & k-2i & l \\ m & n & o-2m & p \end{bmatrix} =$

2. (6 pts) A square matrix A is **orthogonal** if $A^{-1} = A^T$. Prove: If A is orthogonal matrix then $\det A = \pm 1$. Refer to appropriate Facts and Theorems.

3. a) Basis Questions. (5pts) Complete the following definition. $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a **basis** for a vector space $\mathbb{V}$ if:

b) (6pts) Assume $A = \begin{bmatrix} 1 & 3 & -2 & 0 & 0 \\ 2 & 6 & -5 & -2 & -3 \\ 0 & 0 & 5 & 10 & 15 \\ 2 & 6 & 0 & 8 & 18 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 3 & 0 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 8 & 13 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Determine a basis for Nul A.

c) (4pts) Determine a basis for Col A. Explicitly list the vectors.

4. a) (4pts) Is $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^4$? **Justify** your answer.

b) (2pts) Fill in the blank: $\dim \mathbb{P}_4 =$ ____.

c) (6pts) For what values of k, if any, is $B = \left\{ \begin{bmatrix} k \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ 0 \\ k \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$? Note: The same value of k is used in all three locations. Justify your answer. Think! There are at least a couple of ways to do this. One is easiest. Justify your result with appropriate Facts and Theorems.

5. a) (5pts) Complete the following definition: Let $\mathbb{V}$ be a vector space. Then $\mathbb{W}$ is a **subspace** of $\mathbb{V}$ using the same operation as in $\mathbb{V}$ if:

b) (6pts) Let A be a fixed $n \times n$ matrix (A does not change in this problem). Define the subset $\mathbb{E} = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{x}\}$. Carefully determine whether $\mathbb{E}$ is a **subspace** of $\mathbb{R}^n$.

6. All parts of this problem are separate and unrelated.

a) (5pts) Complete the definition: If A is an $m \times n$ matrix, then

$$\text{Nul } A = \left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\}$$

b) (6 pts) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard $n \times n$ matrix A. If $\dim(\text{Nul } A) = 0$, is the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ onto? Prove your answer. (For 1 more point) Is T an isomorphism?

c) (5pts) Assume that $\mathcal{B} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a basis for $\mathbb{P}_2$ where $\mathbf{p}_1 = 1-t$, $\mathbf{p}_2 = 3-2t$, and $\mathbf{p}_3 = 2t+t^2$. Let $\mathbf{q} = 7+t+3t^2$. Determine $[\mathbf{q}]_{\mathcal{B}}$, the $\mathcal{B}$-coordinates of $\mathbf{q}$.

7. a) (5pts) Complete the definition: If $T : \mathbb{V} \to \mathbb{W}$ is a linear transformation between vector spaces, then

$$\ker T =$$

b) (7pts) You may **assume** that $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{q}) = \begin{bmatrix} \mathbf{q}(0) \\ \mathbf{q}(-1) \end{bmatrix}$ is a linear transformation. Determine a basis for $\ker T$. Hint: Let $\mathbf{q}(t) = a + bt + ct^2$.

c) (3pts) $\dim \ker T =$ _______. Is the transformation T in the previous part one-to-one? Explain briefly.

8. a) (7pts) Prove: If A and B are $n \times n$ and $\det(AB) = 5$, then the columns of A are a basis for $\mathbb{R}^n$. Use appropriate definitions, facts, and theorems.

b) (7pts) Assume $T : \mathbb{V} \to \mathbb{W}$ is a linear transformation between vector spaces. Prove: If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a linearly dependent set in $\mathbb{V}$, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly dependent in $\mathbb{W}$. Use appropriate definitions, facts, and theorems.

9. (3pts) Let A be 8×9 matrix. If $\dim(\text{Col}\, A) = 6$, then $\dim(\text{Nul}\, A) =$ _______. Briefly justify your answer.

Problem	1	2	3	4	5	6	7	8	9	Total
Points	8	6	15	12	11	16	15	14	3	100
Score										

Name: ______________________________