1. Compute both the 1-level and the 2-level two-dimensional Haar wavelet transform of the following $4 \times 4$ matrix:

$$
\begin{pmatrix}
2 & -2 & 0 & 0 \\
-2 & -2 & 0 & 0 \\
3 & 1 & 4 & 4 \\
5 & 1 & 0 & 2 \\
\end{pmatrix}
$$

2. A multiresolution analysis based on the 2-level Haar transforms of the matrix in problem 1 would write that matrix as the sum of seven matrixes, $A^2 + (H^2 + V^2 + D^2) + (H^1 + V^1 + D^1)$. Write out this multiresolution analysis explicitly as the sum of seven $4 \times 4$ matrixes.

3. Let $\{b_1, b_2, \ldots, b_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Write $b_i = (b^i_1, b^i_2, \ldots, b^i_n)$, for $i = 1, 2, \ldots, n$. We can associate this basis with the matrix

$$
B = \begin{pmatrix}
b^1_1 & b^1_2 & \ldots & b^1_n \\
b^2_1 & b^2_2 & \ldots & b^2_n \\
\vdots & \vdots & \ddots & \vdots \\
b^n_1 & b^n_2 & \ldots & b^n_n \\
\end{pmatrix}
$$

This matrix is the matrix of the linear transformation $A$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ that maps $b_i$ to the standard basis vector $e_i$, for $i = 1, 2, \ldots, n$. For $x \in \mathbb{R}^n$, the components of $A(x)$ give the expansion of $x$ in the basis $\{b_1, b_2, \ldots, b_n\}$.

Now, we also have the orthonormal basis $\{b_i \otimes b_j | 1 \leq i \leq n, 1 \leq j \leq n\}$ of the vector space $\mathbb{R}^n \otimes \mathbb{R}^n$ and a corresponding linear transformation from $\mathbb{R}^n \otimes \mathbb{R}^n$ to itself. The effect of this transformation on an $n \times n$ matrix $M$ can be computed by applying $A$ to the rows of $M$ and then to the columns of the resulting matrix (or vice versa). Show that this value can also be computed as the matrix product $BMB^T$. [Note: This is not difficult. It’s just a matter of looking at how the matrix product is computed and keeping the definitions straight.]

4. The two-dimensional Discrete Fourier Transform from $\mathbb{C}^n \otimes \mathbb{C}^n$ to $\mathbb{C}^n \otimes \mathbb{C}^n$ can be defined in terms of the basis $\{w_k \otimes w_\ell | 0 \leq k < n, 0 \leq \ell < n\}$. In this definition, the entries in the DFT of an $n \times n$ matrix $M$ are given by the inner products $\langle M, w_k \otimes w_\ell \rangle$.

a) Find the number of operations needed to compute the two-dimensional DFT of $M$ using this definition, giving your answer in the form “some constant times $f(n)$” for a function $f(n)$.

b) Suppose that the DFT of $M$ is computed by applying the one-dimensional DFT to the rows of $M$ and then to the columns of the resulting matrix. Find the number of operations in this case.

c) Suppose that $n$ is a power of 2, so that the FFT can be used to compute the DFT. When the FFT is used in part b), what is the number of operations?

d) Suppose that you want to compute the two-dimensional DFT of a 1024-by-1024 pixel image. ($1024 = 2^{10}$.) Write a paragraph comparing the computational cost of the methods in parts a), b), and c).

5. State the topic of your final project, write a short description of what it will be about, and list several references that you will use in the project.