Math 331, Day 24: The Mean Value Theorem

Reading, Practice

Review Section 3.3 carefully. We are done with differential calculus! Read ahead in Section 3.4. We will begin integration next time.

The Mean Value Theorem. Let $f$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then there is a point $c$ strictly between $a$ and $b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Using the MVT

By the end of class you should be able to prove all of the Results 1 through 5 below.

1. Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Under these hypotheses: Let $c, d \in [a, b]$ with $c < d$. Use the MVT on the interval $[c, d]$ and the given information to determine the relationship between $f(c)$ and $f(d)$ and prove the result.
   (a) If $f'(x) = 0$ for all $x \in (a, b)$, then $f$ is constant on $[a, b]$.
   (b) If $f'(x) > 0$ for all $x \in (a, b)$ and if $c, d \in [a, b]$ with $d < d$, then $f(c) < f(d)$ (that is, $f$ is increasing on $[a, b]$).
   (c) If $f'(x) < 0$ for all $x \in (a, b)$ and if $x, y \in (a, b)$ with $x < y$, then $f(x) > f(y)$ (that is, $f$ is decreasing on $[a, b]$).
   (d) If $f'(x) \neq 0$ for all $x \in (a, b)$, then $f$ is one-to-one on $[a, b]$.

2. Suppose that $f, g$ are continuous on $[a, b]$ and differentiable on $(a, b)$ with $g'(x) = f'(x)$ for all $x \in (a, b)$. Then there is a constant $k$ so that $g(x) = f(x) + k$ on $[a, b]$. Hint: Consider $h(x) = g(x) - f(x)$.

**The next three problems all use the same idea:** Apply the MVT to the correct function $f(t)$ on the interval $[a, x]$, where $a$ is a constant that depends on the question.

3. Use the MVT to prove: If $x \geq 0$, then $\sin x \leq x$. (Assume Calculus I knowledge.) Hint: The result is clearly true if $x = 0$ (right?). So assume $x > 0$. Let $f(t) = \sin t$ on $[0, x]$.

4. Use the MVT to prove Bernoulli’s Inequality: For all $x > 0$ and for all $n \in \mathbb{N}$,

$$(1 + x)^n > 1 + nx.$$

What’s $f(t)$ this time? Note: This can be done by induction, but it is quicker with the MVT.

5. Prove: If $x > 1$, then $\frac{x - 1}{x} < \ln x < x - 1$. What’s $f(t)$ this time?

6. The Cauchy Mean Value Theorem. Suppose that $f$ and $g$ are continuous on the closed, bounded interval $[a, b]$ and are differentiable on $(a, b)$. Then there is a point $c$ strictly between $a$ and $b$ such that

$$(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c).$$

(a) Define the auxiliary function $h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x)$. Show that $h$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

(b) Show that $h(a) = h(b)$.

(c) Apply the MVT to $h$ and show that $c$ is the desired point.

7. Problem 7 on the back, assuming the IVTD which you are proving for Homework.

Over

1. (a) Jack, Kyle  
   (b) Lillie, Alana  
   (c) Nan, Weixiang  
   (d) Liv, Michael, David

2. Jack, Kyle

3. Lillie, Alana  
4. Nan, Weixiang  
5. Liv, Michael, David

6. Jack, Kyle

7. Everyone else
Comments on the Current Assignment

Differentiability on Closed Intervals. We say \( g \) is **differentiable on the closed interval** \([a, b]\) if \( g \) is differentiable at each point in the open interval \((a, b)\) and the appropriate one-sided derivatives exist at \( a \) and \( b \). Specifically

1. \( g \) is differentiable at each \( x \in (a, b) \),
2. \( \lim_{x \to a^+} g(x) - g(a) \) exists (and is denoted by \( g'(a) \)), and \( \lim_{x \to b^-} g(x) - g(b) \) exists (and is denoted by \( g'(b) \)).

Note: All basic derivative rules (e.g., sum, product) carry over to functions differentiable on closed intervals.

Current Homework

5. **Intermediate Value Theorem for Derivatives.** If \( f \) is differentiable on \([a, b]\) and \( f'(a) < k < f'(b) \), then there is a \( c \in (a, b) \) with \( f'(c) = k \). A similar result holds if \( f'(a) > k > f'(b) \). (Note: We cannot apply the IVT because we do not know that \( f' \) is continuous on \([a, b]\).)

(a) Consider the auxiliary function \( g(x) = f(x) - kx \), for \( x \in [a, b] \). Since \( f \) and \( x \) are differentiable on \([a, b]\) it follows that \( g \) is differentiable on \([a, b]\). Show that \( g'(a) < 0 < g'(b) \).

(b) Prove that \( g \) has a minimum point \( c \in [a, b] \).

(c) From part (a), \( 0 < g'(b) = \lim_{x \to b^-} \frac{g(x) - g(b)}{x - b} \). Use the definition of one-sided limit to prove that there exists \( \delta > 0 \) so that if \( -\delta < x - b < 0 \), then \( 0 < \frac{g(x) - g(b)}{x - b} \). Hint: Let \( \varepsilon = g'(b) \).

(d) With this same \( \delta \) prove: If \( -\delta < x - b < 0 \), then \( g(x) < g(b) \). [This shows that \( g(b) \) is NOT the minimum value of \( g \). A similar argument shows that \( g(a) \) is also not the minimum value of \( g \). In other words, \( c \neq a \) and \( c \neq b \).]

(e) So \( c \in (a, b) \). Prove \( g'(c) = 0 \) and then show \( f'(c) = k \).

6. True or False: The Dirichlet function \( D(x) \) is the derivative of some function \( F(x) \) on the interval \([a, b]\). (Is \( D(x) = F'(x) \) for some function \( F \)?) Explain.

On the next assignment—or not

7. **Corollary of IVTFD.** If \( f \) is differentiable on \([a,b]\) and \( f'(x) \neq 0 \) for all \( x \in (a,b) \), then either \( f'(x) < 0 \) for all \( x \in [a,b] \) or \( f'(x) > 0 \) for all \( x \in [a,b] \). (So from Problem 1, \( f \) is either always increasing or always decreasing on \([a,b]\).)
**Math 331, Day 24: The Mean Value Theorem**

**Solutions to In-Class Problems**

1. Suppose that $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$. Under these hypotheses: Let $c,d \in [a,b]$ with $c < d$. Use the MVT on the interval $[c,d]$ and the given information to determine the relationship between $f(c)$ and $f(d)$ and prove the result.

(a) If $f'(x) = 0$ for all $x \in (a,b)$, then $f$ is constant on $[a,b]$.

(b) If $f'(x) > 0$ for all $x \in (a,b)$ and if $c,d \in [a,b]$ with $d < c$, then $f(c) < f(d)$ (that is, $f$ is increasing on $[a,b]$).

(c) If $f'(x) < 0$ for all $x \in (a,b)$ and if $x,y \in (a,b)$ with $x < y$, then $f(x) > f(y)$ (that is, $f$ is decreasing on $[a,b]$).

(d) If $f'(x) \neq 0$ for all $x \in (a,b)$, then $f$ is one-to-one on $[a,b]$. [Hint: Contradiction or contraposition is easy.]

**Proof (a).** Let $c,d \in [a,b]$ with $c < d$. It suffices to show that $f(c) = f(d)$. Since $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$ with $f'(x) = 0$ for all $x \in (a,b)$, then $f$ is continuous on $[c,d]$ and differentiable on $(c,d)$. So the MVT applies to $f$ on $[c,d]$. Therefore, there is a point $z \in [c,d]$ so that

$$f'(z) = \frac{f(d) - f(c)}{d - c}.$$  

Since $f'(z) = 0$, it follows that $f(d) = f(c)$ for all $c,d \in [a,b]$. That is, $f$ is constant.

**Proof (b).** Let $c,d \in [a,b]$ with $c < d$. Show that $f(c) < f(d)$. Since $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$ with $f'(x) > 0$ for all $x \in (a,b)$, then $f$ is continuous on $[c,d]$ and differentiable on $(c,d)$. So the MVT applies to $f$ on $[c,d]$. Therefore, there is a point $z \in [c,d]$ so that

$$\frac{f(d) - f(c)}{d - c} = f'(z) > 0.$$  

Because $c < d$, it follows $f(d) - f(c) > 0$. That is, $f(c) < f(d)$.

**Proof (c).** Let $c,d \in [a,b]$ with $c < d$. Show that $f(c) > f(d)$. Since $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$ with $f'(x) > 0$ for all $x \in (a,b)$, then $f$ is continuous on $[c,d]$ and differentiable on $(c,d)$. So the MVT applies to $f$ on $[c,d]$. Therefore, there is a point $z \in [c,d]$ so that

$$\frac{f(d) - f(c)}{d - c} = f'(z) < 0.$$  

Because $c < d$, it follows $f(d) - f(c) < 0$. That is, $f(c) > f(d)$.

**Proof (d):** Contraposition. Assume $f$ is not one-to-one on $[a,b]$. Then there exist $c,d \in [a,b]$ with $c < d$ such that $f(c) = f(d)$. Since $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$ with $f'(x) \neq 0$ for all $x \in (a,b)$, then $f$ is continuous on $[c,d]$ and differentiable on $(c,d)$. So the MVT applies to $f$ on $[c,d]$. Therefore, there is a point $z \in [c,d]$ so that

$$f'(z) = \frac{f(d) - f(c)}{d - c} = \frac{0}{d - c} = 0.$$  

2. Suppose that $f,g$ are continuous on $[a,b]$ and differentiable on $(a,b)$ with $g'(x) = f'(x)$ for all $x \in (a,b)$. Then there is a constant $k$ so that $g(x) = f(x) + k$ on $[a,b]$. Hint: Consider $h(x) = g(x) - f(x)$.

**Proof.** Let $h(x) = g(x) - f(x)$ for all $x \in [a,b]$. Since $f,g$ are continuous on $[a,b]$ and differentiable on $(a,b)$, then so is $h$. But $h'(x) = g'(x) - f'(x) = 0$ for all $x \in (a,b)$. So by Problem 1(a), $h(x) = k$ is $g(x) - f(x) = k$ and so $g(x) = f(x) + k$.

**Note:** The next three problems all use the same idea: Apply the MVT to the correct function $f(t)$ on the interval $[a,x]$, where $a$ is a constant that depends on the question.
3. Use the MVT to prove: If \( x \geq 0 \), then \( \sin x \leq x \). (Assume Calculus I knowledge.) Hint: The result is clearly true if \( x = 0 \) (right?). So assume \( x > 0 \). Let \( f(t) = \sin t \) on \([0, x]\).

**PROOF.** The result is clearly true if \( x = 0 \) because \( \sin(0) = 0 \). So assume \( x > 0 \). Let \( f(t) = \sin t \) on \([0, x]\) where it is continuous and \( f \) is differentiable on \((0, x)\). By the MVT, there is a point \( c \in [0, x] \) so that

\[
1 \geq \cos(c) = f'(c) = \frac{\sin(x) - \sin(0)}{x - 0} = \frac{\sin(x)}{x}.
\]

Consequently, \( x \geq \sin(x) \).

4. Use the MVT to prove **Bernoulli's Inequality**. For all \( x > 0 \) and for all \( n \in \mathbb{N} \),

\[
(1 + x)^n > 1 + nx.
\]

What's \( f(t) \) this time? Note: This can be done by induction, but it is quicker with the MVT.

**PROOF.** Let \( x > 0 \) and \( n \in \mathbb{N} \). Let \( f(t) = (1 + t)^n \) on \([0, x]\). Since \( f \) is a polynomial, it is continuous and differentiable everywhere. By the MVT, there is a point \( c \in [0, x] \) so that

\[
n(1 + t)^{n-1} = f'(c) = \frac{(1 + x)^n - (1 + 0)^n}{x - 0}.
\]

Consequently,

\[
nx(1 + t)^{n-1} = (1 + x)^n - 1.
\]

Since \( t > 0 \), it follows that \( (1 + t) > 1 \), so \( (1 + t)^{n-1} > 1^{n-1} = 1 \). Therefore,

\[
nx < nx(1 + t)^{n-1} = (1 + x)^n - 1 \quad \text{or} \quad 1 + nx < (1 + x)^n.
\]

5. Prove: If \( x > 1 \), then \( \frac{x - 1}{x} < \ln x < x - 1 \). What's \( f(t) \) this time?

**PROOF.** Let \( f(t) = \ln t \) on \([1, x]\). From Calc 1, \( f \) is continuous and differentiable on \([1, x]\). By the MVT, there is a point \( c \in [1, x] \) so that

\[
\frac{1}{c} = f'(c) = \frac{\ln x - \ln 1}{x - 1} = \frac{\ln x}{x - 1}.
\]

Consequently,

\[
\frac{x - 1}{c} = \ln x.
\]

Since \( 1 < c < x \), it follows that

\[
\frac{x - 1}{x} < \frac{x - 1}{c} = \ln x < \frac{x - 1}{1} = x - 1.
\]

6. The **Cauchy Mean Value Theorem**. Suppose that \( f \) and \( g \) are continuous on the closed, bounded interval \([a, b]\) and are differentiable on \((a, b)\). Then there is a point \( c \) strictly between \( a \) and \( b \) such that

\[
(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c).
\]

**PROOF.** Define the auxiliary function \( h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x) \). Since \( f, g \) are continuous on \([a, b]\) and differentiable on \((a, b)\), so are the constant multiples \((g(b) - g(a))f(x)\) and \((f(b) - f(a))g(x)\) and consequently, so is their difference \( h(x) \). Next

\[
h(a) = (g(b) - g(a))f(a) - (f(b) - f(a))g(a) = g(b)f(a) - f(b)g(a) = (g(b) - g(a))f(b) - (f(b) - f(a))g(b)
\]

so \( h(a) = h(b) \). Applying the MVT to \( h \), there is point \( c \in [a, b] \) so that

\[
h'(c) = \frac{h(b) - h(a)}{b - a} = \frac{0}{b - a} = 0.
\]

But then

\[
h'(c) = (g(b) - g(a))f'(c) - (f(b) - f(a))g'(c) = 0
\]

and the result follows.
7. **Corollary of IVTFD.** If $f$ is differentiable on $[a,b]$ and $f'(x) \neq 0$ for all $x \in (a,b)$, then either $f'(x) \geq 0$ for all $x \in [a,b]$ or $f'(x) \leq 0$ for all $x \in [a,b]$. (So from Problem 1, $f$ is either always increasing or always decreasing on $[a,b]$.)

**Proof (Contradiction).** Suppose that there exist $c,d \in [a,b]$ with $f'(c) < 0$ and $f'(d) > 0$. Since $f$ is differentiable (and so continuous) on $[a,b]$, then $f$ is differentiable and continuous on $[c,d]$, so the IVTFD applies there. Since $f'(c) < 0 < f'(d)$, there is point $z \in [c,d]$ so that $f'(z) = 0$. Since $[c,d] \subseteq [a,b]$, it follows that $z \in [a,b]$. This contradicts that $f'(x) \neq 0$ for all $x \in (a,b)$.