Sets and Functions

Key Points

- terms and concepts: set, element; empty set, power set; subset, disjoint sets
- notation
- defining sets
- set operations
- quantifiers and sets

Terms and Concepts

- a *set* is a collection of *elements*
  - an element can be anything, including another set
  - set is completely defined by its elements
  - the order in which the elements are listed is not important
  - no duplicates
- a set can be defined by listing its elements: { *a*, *b*, *c*, … }
  - { *a*, *b*, *c* } does not necessarily require three different elements unless "*a*, *b*, *c* are distinct" is specified
  - *empty set* { } or ∅ contains no elements
- a set can be defined by predicates:
  - { x | P(x) } requires that the domain of discourse be a set, which is problematic because there is no set of all sets

$$\{x \in X \mid P(x)\}$$
$$\{x \mid x \in X \land P(x)\}$$

Terms and Concepts

- two sets A, B are *equal* if they contain the same elements
- two sets A, B are *disjoint* if they have no elements in common
- a set A is a *subset* of a set B if everything in A is also in B
  - A is a *proper subset* if there's at least one thing in B that isn't in A
  - the empty set is a subset of any set
- the *power set* of A is the set of all subsets of A

For example, if $A = \{a, b\}$, then the subsets of A are the empty set, $\{a\}$, $\{b\}$, and $\{a, b\}$, so the power set of A is set given by

$$\mathcal{P}(A) = \{\, \emptyset,\ \{a\},\ \{b\},\ \{a, b\} \,\}.$$

  - the power set of the empty set is { ∅ }
    - { ∅ } ≠ { }

Set Operations

- *union* – elements in either set (or both)
- *intersection* – elements in both sets
- *difference* – elements in A that aren't in B

Suppose that $A = \{a, b, c\}$, that $B = \{b, d\}$, and that $C = \{d, e, f\}$. Then we can apply the definitions of union, intersection, and set difference to compute, for example, that:

$$A \cup B = \{a, b, c, d\} \qquad A \cap B = \{b\} \qquad A \smallsetminus B = \{a, c\}$$
$$A \cup C = \{a, b, c, d, e, f\} \qquad A \cap C = \emptyset \qquad A \smallsetminus C = \{a, b, c\}$$

Notation

Notation	Definition
$a \in A$	a is a member (or element) of A
$a \notin A$	$\neg(a \in A)$, a is not a member of A
$\emptyset$	the empty set, which contains no elements
$A \subseteq B$	A is a subset of B, $\forall x(x \in A \rightarrow x \in B)$
$A \subsetneq B$	A is a proper subset of B, $A \subseteq B \wedge A \neq B$
$A \supseteq B$	A is a superset of B, same as $B \subseteq A$
$A \supsetneq B$	A is a proper superset of B, same as $B \supsetneq A$
$A = B$	A and B have the same members, $A \subseteq B \wedge B \subseteq A$
$A \cup B$	union of A and B, $\{x \mid x \in A \vee x \in B\}$
$A \cap B$	intersection of A and B, $\{x \mid x \in A \wedge x \in B\}$
$A \smallsetminus B$	set difference of A and B, $\{x \mid x \in A \wedge x \notin B\}$
$\mathcal{P}(A)$	power set of A, $\{X \mid X \subseteq A\}$

Figure 2.1: Some of the notations that are defined in this section. A and B are sets, and a is an entity.

also { }

$(\forall x \in A)(P(x))$ is true iff P(a) for every element *a* of the set A

$(\exists x \in A)(P(x))$ is true iff there is some element *a* of the set A for which P(a) is true

2. Compute $A \cup B$, $A \cap B$, and $A \smallsetminus B$ for each of the following pairs of sets
- **a)** $A = \{a, b, c\}$, $B = \emptyset$
- **b)** $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 10\}$
- **c)** $A = \{a, b\}$, $B = \{a, b, c, d\}$
- **d)** $A = \{a, b, \{a, b\}\}$, $B = \{\{a\}, \{a, b\}\}$

3. Recall that $\mathbb{N}$ represents the set of natural numbers. That is, $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. Let $X = \{n \in \mathbb{N} \mid n \geq 5\}$, let $Y = \{n \in \mathbb{N} \mid n \leq 10\}$, and let $Z = \{n \in \mathbb{N} \mid n \text{ is an even number}\}$. Find each of the following sets:
- **a)** $X \cap Y$ **b)** $X \cup Y$ **c)** $X \smallsetminus Y$ **d)** $\mathbb{N} \smallsetminus Z$
- **e)** $X \cap Z$ **f)** $Y \cap Z$ **g)** $Y \cup Z$ **h)** $Z \smallsetminus \mathbb{N}$

4. Find $\mathcal{P}(\{1, 2, 3\})$. (It has eight elements.)

6. Since $\mathcal{P}(A)$ is a set, it is possible to form the set $\mathcal{P}(\mathcal{P}(A))$. What is $\mathcal{P}(\mathcal{P}(\emptyset))$? What is $\mathcal{P}(\mathcal{P}(\{a, b\}))$? (It has sixteen elements.)

5. Assume that a and b are entities and that $a \neq b$. Let A and B be the sets defined by $A = \{\, a, \{b\}, \{a, b\}\,\}$ and $B = \{\, a, b, \{a, \{b\}\}\,\}$. Determine whether each of the following statements is true or false. Explain your answers.
- **a)** $b \in A$ **b)** $\{a, b\} \subseteq A$ **c)** $\{a, b\} \subseteq B$
- **d)** $\{a, b\} \in B$ **e)** $\{a, \{b\}\} \in A$ **f)** $\{a, \{b\}\} \in B$

8. If A is any set, what can you say about $A \cup A$? About $A \cap A$? About $A \smallsetminus A$? Why?

9. Suppose that A and B are sets such that $A \subseteq B$. What can you say about $A \cup B$? About $A \cap B$? About $A \smallsetminus B$? Why?

10. Suppose that A, B, and C are sets. Show that $C \subseteq A \cap B$ if and only if $(C \subseteq A) \wedge (C \subseteq B)$.

11. Suppose that A, B, and C are sets, and that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.

12. Suppose that A and B are sets such that $A \subseteq B$. Is it necessarily true that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$? Why or why not?