One-to-One Correspondence

- a *one-to-one correspondence* between sets *A* and *B* means every element of *A* is paired with an element of *B* and every element of *B* is paired with an element of *A*
  - demonstrates that two sets have the same number of elements
  - counting establishes a one-to-one correspondence between a set with *n* elements and the set of numbers { 1, 2, …, *n* }

Theorem 2.6. *For each $n \in \mathbb{N}$, let N_n be the set $N_n = \{0, 1, \ldots, n-1\}$. If $n \neq m$, then there is no bijective function from N_m to N_n.*

  - there can't be a one-to-one correspondence between sets of a different size

Finite and Infinite

- a set *A* is *finite* is there is a one-to-one correspondence between *A* and N_n for some natural number *n*
- a set *A* is *infinite* if there is no such *n*

Cardinality

- for a finite set *A*, the *cardinality* of *A*, written |*A*|, is the number of elements in *A*

Theorem 2.7. *A finite set with cardinality n has 2^n subsets.*

Theorem 2.8. *Let A and B be finite sets. Then*

- $|A \times B| = |A| \cdot |B|$.
- $|A \cup B| = |A| + |B| - |A \cap B|$.
- *If A and B are disjoint then $|A \cup B| = |A| + |B|$.*
- $|A^B| = |A|^{|B|}$.
- $|\mathcal{P}(A)| = 2^{|A|}$.

A^B is the set of functions $A \to B$

1. Suppose that A, B, and C are finite sets which are pairwise disjoint. (That is, $A \cap B = A \cap C = B \cap C = \emptyset$.) Express the cardinality of each of the following sets in terms of $|A|$, $|B|$, and $|C|$. Which of your answers depend on the fact that the sets are pairwise disjoint?

a) $\mathcal{P}(A \cup B)$ **b)** $A \times (B^C)$ **c)** $\mathcal{P}(A) \times \mathcal{P}(C)$
d) $A^{B \times C}$ **e)** $(A \times B)^C$ **f)** $\mathcal{P}(A^B)$
g) $(A \cup B)^C$ **h)** $(A \cup B) \times A$ **i)** $A \times A \times B \times B$

2. Suppose that A and B are finite sets which are not necessarily disjoint. What are all the possible values for $|A \cup B|$?

3. Let's say that an "identifier" consists of one or two characters. The fist character is one of the twenty-six letters (A, B, . . . , C). The second character, if there is one, is either a letter or one of the ten digits (0, 1, . . . , 9). How many different identifiers are there? Explain your answer in terms of unions and cross products.

4. Suppose that there are five books that you might bring along to read on your vacation. In how many different ways can you decide which books to bring, assuming that you want to bring at least one? Why?

Infinities

- a set A is *countably infinite* if there is a one-to-one correspondence between $\mathbb{N}$ and A
- a set A is *countable* if it is either finite or countably infinite
  - it is possible in principle to make a list of the elements of A
- a set A is *uncountable* otherwise
  - it is impossible to make a list of the elements of A

Infinities

- $\mathbb{Z}$ (integer) is countably infinite

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0  1  -1  2  -2  3  -3  ...
0  1   2  3   4  5   6  ...
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- $\mathbb{Q}$ (rationals) is countably infinite

$$\left(\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \frac{1}{5}, \frac{5}{1}, \frac{1}{6}, \frac{2}{5}, \ldots\right)$$

10. Show that the set $\mathbb{N} \times \mathbb{N}$ is countable.

- use a diagonalization argument

$$\begin{array}{cccc} (0,0) & (0,1) & (0,2) & \cdots \\ (1,0) & (1,1) & (1,2) & \cdots \\ (2,0) & (2,1) & (2,2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

$(0,0), (0,1), (1,0), (0,2), (1,1), (2,0), (0,3), (1,2), (2,1), (3,0), \ldots$

a) Suppose that A and B are countably infinite sets. Show that $A \cup B$ is countably infinite.

Infinities

- $\mathbb{R}$ (reals) is not countably infinite

$0.\mathbf{9}0398937249879561297927654857945\ldots$
$0.1\mathbf{2}349342094059875980239230834549\ldots$
$0.22\mathbf{4}000432984362347093232799895 79\ldots$
$0.500\mathbf{0}0000000000000000000000000000\ldots$
$0.7774\mathbf{3}449234234876990120909480009\ldots$
$0.77755\mathbf{5}55588888888949888898000011 1\ldots$
$0.123456\mathbf{7}888888888888888888800000000\ldots$
$0.3483544\mathbf{0}00984871271212394032057 7\ldots$
$0.93473244\mathbf{4}47900498340999990948900\ldots$

for any such list, a new number not in the listed can be constructed by picking a number not in bold for each column

- $\mathbb{R}\backslash\mathbb{Q}$ (irrationals) is not countably infinite

Theorem 2.9. *Suppose that X is an uncountable set, and that K is a countable subset of X. Then the set $X \smallsetminus K$ is uncountable.*

Infinities

Theorem 2.11. *Let X be any set. Then there is no one-to-one correspondence between X and $\mathcal{P}(X)$.*

  - for finite sets, $|\mathcal{P}(X)| = 2^{|X|} > |X|$
  - the "larger" relationship holds for infinite sets too

- can construct an infinite series of increasingly larger infinities with $\mathbb{R}$, $\mathcal{P}(\mathbb{R})$, $\mathcal{P}(\mathcal{P}(\mathbb{R}))$, $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{R})))$, …