Identify the language generated by the following context-free grammar.

$$S \longrightarrow aaSb$$
$$S \longrightarrow \epsilon$$

Answer: $\{a^{2n}b^n \mid n \geq 0\}$, or as followed by bs with twice as many as as bs.

Discussion: It can be possible to just reason about the rules, but starting to methodically generate the strings of the language can help reveal patterns that can then be concisely described.

Start with the start symbol: S.

There are two productions that can be applied: $S \to aaSb$ and $S \to \epsilon$. Given these rules, we can conclude

$$S \Longrightarrow aaSb \quad (\text{using } S \to aaSb)$$
$$S \Longrightarrow \epsilon \quad (\text{using } S \to \epsilon)$$

ϵ doesn't contain any terminal symbols (it doesn't contain any symbols at all), so it is one of the strings generated by this grammar.

Continue with $aaSb$ — this contains an S, and there are two productions that can be applied.

$$aaSb \Longrightarrow aaaaSbb \quad (\text{using } S \to aaSb)$$
$$aaSb \Longrightarrow aab \quad (\text{using } S \to \epsilon)$$

aab is a second string in the language. Keep going:

$$aaaaSbb \Longrightarrow aaaaaaSbbb \quad (\text{using } S \to aaSb)$$
$$aaaaSbb \Longrightarrow aaaabb \quad (\text{using } S \to \epsilon)$$

$aaaabb$ is a third string in the language.

At some point, it should become clear that we are getting strings of the form $a^{2n}b^n$ for $n \geq 0$. (This can also be reasoned — each application of the rule $S \longrightarrow aaSb$ results in two as and one b, and the final step using the rule $S \longrightarrow \epsilon$ doesn't change the number of as and bs.)

Identify the language generated by the following context-free grammar.

$$S \longrightarrow aSb$$
$$S \longrightarrow aaSb$$
$$S \longrightarrow \epsilon$$

Answer: $\{a^m b^n \mid n \geq 0 \text{ and } n \leq m \leq 2n\}$, or as followed by bs where there are at least as many as as bs but not more than twice as many.

Discussion: Given the previous example, we might be able to reason directly that there's always one S and depending on the choice of production when we replace S, we get one b and one or two as — that means at most we'll get twice as many as (if $S \longrightarrow aaSb$ is always used until the final step) and at least the same number of as as bs (if $S \longrightarrow aSb$ is always used until the final step).

But it is also possible to use the same enumeration approach —

$$S \Longrightarrow aSb \quad (\text{using } S \to aSb)$$
$$S \Longrightarrow aaSb \quad (\text{using } S \to aaSb)$$
$$S \Longrightarrow \epsilon \quad (\text{using } S \to \epsilon)$$

Then continue with both aSb and $aaSb$:

$$aSb \Longrightarrow aaSbb \quad (\text{using } S \to aSb)$$
$$aSb \Longrightarrow aaaSbb \quad (\text{using } S \to aaSb)$$
$$aSb \Longrightarrow ab \quad (\text{using } S \to \epsilon)$$

$$aaSb \Longrightarrow aaaSbb \quad (\text{using } S \to aSb)$$
$$aaSb \Longrightarrow aaaaSbb \quad (\text{using } S \to aaSb)$$
$$aaSb \Longrightarrow aab \quad (\text{using } S \to \epsilon)$$

And so forth until the pattern becomes apparent.

Identify the language generated by the following context-free grammar.

$$S \longrightarrow TS$$
$$S \longrightarrow \epsilon$$
$$T \longrightarrow aTb$$
$$T \longrightarrow \epsilon$$

Answer: the set of strings where each maximal stretch of as is followed by the same number of bs — $a^m b^m a^n b^n a^p b^p \ldots$ where $m, n, p, \ldots \geq 0$.

Discussion: As the grammars get larger, methodically deriving enough strings to spot patterns gets more difficult (and more time-consuming). So we might try reasoning.

Consider first the T rules — these have a similar pattern to the rules in the previous examples, and we can observe that starting with a T ultimately results in a string of the form $a^n b^n$ for $n \geq 0$ (and the intermediate form is $a^n T b^n$ — only one non-terminal, and it is a T).

Now, consider S. Starting from S results in zero or more Ts: $S \Longrightarrow TS \Longrightarrow TTS \Longrightarrow TTTS$ and so forth. Since each T then turns into $a^n b^n$, the result is the set of strings where each maximal stretch of as is followed by the same number of bs. ("Maximal" is this context means that the string of as can't get bigger by adding more to it — it is bounded on either side by the beginning of the string or a b.)

Find a context-free grammar that generates the language $\{a^n b^m \mid n \geq m > 0\}$.

Answer:

$$S \longrightarrow aSb$$
$$S \longrightarrow aS$$
$$S \longrightarrow ab$$

Discussion: This language is strings of as followed by bs, where there are at least as many as as bs.

Let's start with a grammar that produces strings as followed by bs where there are the same number of as and bs — that's not so different from parts of the examples above.

$$S \longrightarrow aSb$$
$$S \longrightarrow \epsilon$$

But this isn't quite right for our needs — $m > 0$, so there needs to be at least one b (and thus also at least one a). $S \Longrightarrow \epsilon$ is the problem here, so substitute the actual shortest legal string for ϵ.

$$S \longrightarrow aSb$$
$$S \longrightarrow ab$$

Check that this produces $\{a^m b^m \mid m > 0\}$.

So now we need a way to add extra as. Make sure they go before any bs. (Why wouldn't $S \longrightarrow Sa$ work?)

$$S \longrightarrow aSb$$
$$S \longrightarrow aS$$
$$S \longrightarrow ab$$

Try methodically generating the strings that this produces, for practice with that skill and well as a check on the reasoning done.