Find a context-free grammar that generates the language $\{a^n b^m \mid n \geq 0 \land m = n + 1\}$.

Answer:

$$\begin{aligned} S &\longrightarrow aSb \\ S &\longrightarrow b \end{aligned}$$

Discussion: This language consists of strings of as followed by bs, with one more b than a.

Let's start with a grammar that produces strings with as followed by bs where there are the same number of as and bs.

$$\begin{aligned} S &\longrightarrow aSb \\ S &\longrightarrow \epsilon \end{aligned}$$

Since we only want one extra b, it should be added at the end.

$$\begin{aligned} S &\longrightarrow aSb \\ S &\longrightarrow b \end{aligned}$$

This no longer allows ϵ, but that's OK — there can be 0 as, but one more b than a means there has to be at least 1 b.

Find a context-free grammar that generates the language $\{a^n b^m c^n \mid n, m \in \mathbb{N}\}$.

Answer:

$$
\begin{aligned}
S &\longrightarrow aSc \\
S &\longrightarrow B \\
B &\longrightarrow bB \\
B &\longrightarrow \epsilon
\end{aligned}
$$

Discussion: This language consists of strings of as followed by bs followed by cs, with the same number of as and cs.

Let's start with a grammar that produces strings with as followed by cs where there are the same number of as and cs.

$$
\begin{aligned}
S &\longrightarrow aSc \\
S &\longrightarrow \epsilon
\end{aligned}
$$

To generate a string of bs

$$
\begin{aligned}
B &\longrightarrow bB \\
B &\longrightarrow \epsilon
\end{aligned}
$$

To switch from replacing Ss (and generating as and cs) to replace B (and generating bs), add a rule $S \longrightarrow B$.

Putting these all together:

$$
\begin{aligned}
S &\longrightarrow aSc \\
S &\longrightarrow \epsilon \\
S &\longrightarrow B \\
B &\longrightarrow bB \\
B &\longrightarrow \epsilon
\end{aligned}
$$

Check that this works as desired. $S \longrightarrow \epsilon$ allows ϵ, which is in the language. (Also $S \Longrightarrow B \Longrightarrow \epsilon$, so $S \longrightarrow \epsilon$ isn't actually needed.) ac can be generated, as can abc.