To show that an argument is valid, you need to show that the conjunction of the premises implying the conclusion is a tautology. For #1, that means showing that $(p \to q) \land (\neg p \to q) \to q$ is a tautology.

#1 also asked for an explanation of why the argument makes sense — this isn't asking for only an English translation of the notation, but rather for an understanding of what that combination of premises means. (p and $\neg p$ are the only two possibilities, so if q is true in both cases, it's always be true.)

For formal proofs (#2, #3), common problems included not citing the rule justifying each step and combining multiple steps into one. In a few cases, invalid deductions were made, though this generally only occurred in cases where justifications weren't given — so be sure to include those justifications!

For #3, write down the argument in formal logic even if it is invalid, and then either give a proof or explain why it isn't valid — in some cases, only part of this was done.

When translating English to formal logic, you may have a choice between simple propositions or predicates. In #3a, all of the statements are about a single entity ("this card") rather than something that could apply to multiple entities of some type. Defining propositions R for "this card is red", K for "this card is a king", and H for "this card is a heart" is reasonable.

In #3b, however, some of the statements are about "a math major" (so they apply to any person who is a math major) and some are about a specific person (Alice). So a direct translation of the problem statement would be to define predicates $M(x)$ for "person x is a math major", $A(x)$ for "person x takes Abstract Algebra", $F(x)$ for "person x takes Foundations of Analysis", and $G(x)$ for "person x knows about Galois". Then the argument would be written:

$$\begin{array}{l}
\forall x(M(x) \to A(x) \land F(x)) \\
\forall x(A(x) \to G(x)) \\
M(\text{Alice}) \\
\neg F(\text{Alice}) \\
\hline
\therefore G(\text{Alice})
\end{array}$$

With variables like x, quantifiers are needed — $M(x)$ isn't by itself a proposition because x isn't a specific entity, so just writing something like $M(x) \to A(x) \land F(x)$ isn't quite right.

There are two other ways to establish the validity of the conclusion that Alice knows about Galois. One would be to use that Alice is a member of the domain of discourse and make the propositions specifically about Alice: propositions M, A, F, G are "Alice is a math major", "Alice takes Abstract Algebra", etc. Another would be to make the argument about people in general — anyone who is a math major and doesn't take

Foundations of Analysis knows about Galois — and then use that Alice is a member of the domain of discourse to make the conclusion apply to her. Then propositions M, A, F, G are "someone is a math major", "someone takes Abstract Algebra", etc. In both cases, the argument would be written:

$$\begin{array}{l} M \rightarrow A \wedge F \\ A \rightarrow G \\ M \\ \neg F \\ \hline \therefore G \end{array}$$

While the latter options are not technically direct translations of the statement given in the problem, they were accepted for full credit for the problem.

For #4–8, write informal arguments — you don't need steps written in a numbered list with a rule cited in every case.

However, don't be *too* informal. There still needs to be a careful chain of reasoning, and you still need to write what you are trying to show (only if your proof goes immediately below the problem statement can you omit that statement) and what you are assuming. You also need to state what you are showing if you aren't doing a direct proof. Connect the dots — don't take too many deductive steps at once, provide justifications when bringing in external facts not well-known to the expected domain of readers of your proof, and explain why the last step in your chain of reasoning shows what you are trying to prove.

Constructing a proof should be like writing a paper, and the proof you hand in should be like the paper you hand in — well-organized and clear, with complete sentences (or mathematical statements). Expect to write a draft as you are working out the ideas and flow of the proof, then revise into a more final form.

For example, simply writing

$$n^2 = (4k)^2 = 16k^2 = 4(4k^2)$$

is not a complete proof for #4a. It captures the gist of the argument, but doesn't connect the dots for the reader and lacks a polished style:

Proposition. *If n is an integer and n is divisible by 4, then n^2 is divisible by 4.*

Proof. Let n be an integer divisible by 4. Then $n = 4k$ for an integer k and $n^2 = (4k)^2 = 16k^2 = 4(4k^2)$. $4k^2$ is an integer because it involves a product of integers, so n^2 is divisible by 4. $\square$

Counterexamples don't need to be written as proofs, formal or otherwise. Just give the example, with an explanation of why it makes the statement false.

Proposition. *If n is an integer and n^2 is divisible by 4, then n is divisible by 4.*

Proof. Let $n = 2$. Then $n^2 = 4$ is divisible by 4, but n is not. Thus, this proposition is false. □

To prove an implication $p \rightarrow q$ using a direct proof, assume p and deduce q.

An indirect proof of $p \rightarrow q$ means to use the contrapositive $\neg q \rightarrow \neg p$: assume $\neg q$ and deduce $\neg p$.

Proving a proposition by contradiction means to assume the opposite and deduce a demonstrably false statement. For $p \rightarrow q$, that means to assume $\neg(p \rightarrow q) \equiv p \wedge \neg q$ and then deduce a false statement (which may be $\neg p$ or could be something else). Note that you must actually *use* $\neg q$ in your string of deductions — it is not a proof by contradiction to state that you are assuming p and $\neg q$ but then proceed with deductions based on only p. (If you get to q, you actually have a direct proof of $p \rightarrow q$; if you get to $\neg q$ (the implication $p \rightarrow q$ is false, not true; and you shouldn't get to $\neg p$ having only assumed p.)

For #6, make sure you show both directions — iff means that you need both $p \rightarrow q$ and $q \rightarrow p$.

Also, n not divisible by 3 means that $n = 3k + 1$ or $n = 3k + 2$ for an integer k — in a number of cases, $n = 3k + 1$ was considered but not $n = 3k + 2$. To handle an "or" like this, show both cases — that whatever works for both $n = 3k + 1$ and $n = 3k + 2$. That way the result is there no matter which situation applies for a given n.

When applying definitions to conclude something, don't forget about the conditions that apply to the definition. For example, in #9, a number of solutions proceeded as follows:

Proposition. *If x is an irrational number and r is a non-zero rational number, xr is an irrational number.*

Flawed proof. Assume xr is rational, which means that $xr = a/b$ for integers a, b where $b \neq 0$. Then $x = a/(br)$ and so x is rational. □

The problem is that concluding that x is rational requires that a and br to be integer and $br \neq 0$ — but all we know about r is that it is rational (and non-zero), which isn't enough to ensure that br is integer.

With a proof by contradiction, we are often trying to deduce $\neg p$ from $\neg q$ in order to show the contradiction. In this case, though p is two things — x is irrational and r is a non-zero rational number — so showing the opposite of either is sufficient to show $\neg p$.

Since "non-zero rational" has a useful definition, let's use that and try to show that x is rational.

Proof. Let x be an irrational number and r be a non-zero rational number. Assume xr is rational, which means that $xr = a/b$ for integers a, b where $b \neq 0$. Also $r = c/d$ for integers c, d where $c, d \neq 0$ ($c \neq 0$ because $r \neq 0$. Then $xr = x(c/d) = a/b$, so $x = (ad)/(bc)$ where ad and bc are integer (product of two integers) and $bc \neq 0$ $(b, c \neq 0)$. Thus x is rational.

This contradicts that x is irrational, so xr cannot be rational i.e. it is irrational. $\square$