Prove that $n^3 + 3n^2 + 2n$ is divisible by 3 for all natural numbers n.

Answer:

Proof. Let $P(n)$ be the statement that n^3+3n^2+2n is divisible by 3. We use induction to show that $P(n)$ is true for all $n \geq 0$.

Base case: Consider the case $n = 0$. Then $n^3 + 3n^2 + 2n = 0^3 + 3(0)^2 + 2(0) = 0$, which is divisible by 3.

Inductive case: Let $k > 0$ be an arbitrary number and assume that $P(k)$ is true, meaning that $k^3 + 3k^2 + 2k$ is divisible by 3. We want to show that $P(k + 1)$ is true, in other words, show that $(k + 1)^3 + 3(k + 1)^2 + 2(k + 1)$ is divisible by 3.

$$\begin{aligned}(k+1)^3 + 3(k+1)^2 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 + 2k + 2 \\ &= k^3 + 6k^2 + 11k + 3 \\ &= (k^3 + 3k^2 + 2k) + (3k^2 + 9k + 3)\end{aligned}$$

By the induction hypothesis, $k^3 + 3k^2 + 2k$ is divisible by 3 so there is an integer c such that $k^3 + 3k^2 + 2k = 3c$.

$$\begin{aligned}(k+1)^3 + 3(k+1)^2 + 2(k+1) &= 3c + (3k^2 + 9k + 3) \\ &= 3c + 3(k^2 + 3k + 1) \\ &= 3(c + k^2 + 3k + 1)\end{aligned}$$

Since c and k are integers, $c + k^2 + 3k + 1$ is integer and thus $P(k + 1)$ is divisible by 3. □

Discussion:

$P(n)$ is that $n^3 + 3n^2 + 2n$ is divisible by 3. To prove this by induction, we need to show $P(0) \wedge \forall k(P(k) \to P(k + 1))$.

Start with the base case, P(0). Plugging in $n = 0$ yields $0^3 + 3(0)^2 + 2(0) = 0$, which is divisible by 3.

Next, the induction case. Let k be any natural number and assume P(k), that is, that $k^3 + 3k^2 + 2k$ is divisible by 3. Now we need to show $P(k + 1)$.

A good starting point is to determine what $P(k + 1)$ actually is — plug $k + 1$ in for n: $P(k + 1)$ is the statement that $(k + 1)^3 + 3(k + 1)^2 + 2(k + 1)$ is divisible by 3.

Now let's try to make use of the induction hypothesis: $k^3 + 3k^2 + 2k$ is divisible by 3. Let's see if we can rearrange $(k + 1)^3 + 3(k + 1)^2 + 2(k + 1)$ a bit so that this can be used...

$$\begin{aligned}(k+1)^3 + 3(k+1)^2 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 + 2k + 2 \\ &= k^3 + 6k^2 + 11k + 3 \\ &= (k^3 + 3k^2 + 2k) + (3k^2 + 9k + 3)\end{aligned}$$

"x divisible by 3" means that there is an integer c such that $x = 3c$, and the induction hypothesis is that $k^3 + 3k^2 + 2k$ is divisible by 3 —

$$\begin{aligned}(k+1)^3 + 3(k+1)^2 + 2(k+1) &= 3c + (3k^2 + 9k + 3)\\ &= 3c + 3(k^2 + 3k + 1)\\ &= 3(c + k^2 + 3k + 1)\end{aligned}$$

c is an integer by the definition of "divisible" and $k^2 + 3k + 1$ is an integer because k is an integer, so this last line meets the definition for "$(k+1)^3 + 3(k+1)^2 + 2(k+1)$ is divisible by 3".

Prove that

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$$

for any natural number n and for any real number r such that $r \neq 1$.

Answer:

Proof. Let $P(n)$ be the statement that

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$$

where $r \neq 1$. We use induction to show that $P(n)$ is true for all $n \geq 0$.

Base case: Consider the case $n = 0$. Then

$$\begin{aligned}\sum_{i=0}^{n} r^i &= \sum_{i=0}^{0} r^i\\ &= r^0\\ &= 1\end{aligned}$$

and

$$\begin{aligned}\frac{1 - r^{n+1}}{1 - r} &= \frac{1 - r^1}{1 - r}\\ &= \frac{1 - r}{1 - r}\\ &= 1\end{aligned}$$

and thus $P(0)$ is true.

Inductive case. Let $k > 0$ be an arbitrary number and assume that $P(k)$ is true, meaning that $\sum_{i=0}^{k} r^i = \frac{1 - r^{k+1}}{1 - r}$. We want to show that $P(k+1)$ is true, in other words, show that $\sum_{i=0}^{k+1} r^i = \frac{1 - r^{k+2}}{1 - r}$.

$$\begin{aligned}\sum_{i=0}^{k+1} r^i &= \sum_{i=0}^{k} r^i + r^{k+1} \\ &= \frac{1-r^{k+1}}{1-r} + r^{k+1} \\ &= \frac{1-r^{k+1}+r^{k+1}(1-r)}{1-r} \\ &= \frac{1-r^{k+1}+r^{k+1}-r^{k+2}}{1-r} \\ &= \frac{1-r^{k+2}}{1-r}\end{aligned}$$

Thus $\sum_{i=0}^{k+1} r^i = \frac{1-r^{k+2}}{1-r}$.

□

Discussion:

$P(n)$ is that $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$. To prove this by induction, we need to show $P(0) \wedge \forall k(P(k) \rightarrow P(k+1))$.

Start with the base case, P(0). Plugging in $n = O$ yields $\sum_{i=0}^{0} r^i = r^0 = 1$. On the other side of the equation, $\frac{1-r^{0+1}}{1-r} = \frac{1-r}{1-r} = 1$. Both sides are 1, $P(0)$ is true.

Next, the induction case. Let k be any natural number and assume $P(k)$, that is,

$$\sum_{i=0}^{k} r^i = \frac{1-r^{k+1}}{1-r}$$

Now we need to show $P(k+1)$:

$$\sum_{i=0}^{k+1} r^i = \frac{1-r^{k+2}}{1-r}$$

Let's try to make use of the induction hypothesis: $\sum_{i=0}^{k} r^i = \frac{1-r^{k+1}}{1-r}$. Let's see if we can rearrange $\sum_{i=0}^{k+1} r^i$ a bit to see how this can be used. Recall the definition of the $\sum$ notation:

$$\sum_{i=0}^{n} f(i) = f(0) + f(1) + f(2) + \ldots + f(n)$$

So

$$\begin{aligned}\sum_{i=0}^{k+1} r^i &= r^0 + r^1 + r^2 + \ldots + r^k + r^{k+1} \\ &= \sum_{i=0}^{k} r^i + r^{k+1}\end{aligned}$$

Now we can use the induction hypothesis:

$$\begin{aligned}
\sum_{i=0}^{k+1} r^i &= \sum_{i=0}^{k} r^i + r^{k+1} \\
&= \frac{1-r^{k+1}}{1-r} + r^{k+1} \\
&= \frac{1-r^{k+1}+r^{k+1}(1-r)}{1-r} \\
&= \frac{1-r^{k+1}+r^{k+1}-r^{k+2}}{1-r} \\
&= \frac{1-r^{k+2}}{1-r}
\end{aligned}$$

...and the desired result:

$$\sum_{i=0}^{k+1} r^i = \frac{1-r^{k+2}}{1-r}$$