Every prime number greater than 2 is odd.

Answer:

Proof. Let x be a prime number greater than 2. Since x is prime and greater than 2, x is divisible only by 1 and x. But if x is greater than 2, and is only divisible by 1 and itself, it is not divisible by 2. Thus x is odd. $\square$

Discussion:

A good way to get started on a proof if you don't have any ideas is to identify the relevant pattern(s), and to make use of the premises and relevant definitions.

Starting with patterns, "every prime number greater than 2 is odd" fits the pattern $\forall x P(x)$, and a tactic for "for all" statements is:

> Let x be an arbitrary number. We want to show that if x is a prime number greater than 2, x is odd.

Continuing with patterns, "If x is a prime number greater than 2, x is odd" has the form $p \to q$, and a tactic for implications is to assume p and show q.

> Assume that x is a prime number greater than 2.

Now what? There are some relevant definitions in the book: *an integer $n > 1$ is prime if it is divisible by exactly two positive numbers, 1 and itself* and *an integer is even if and only if it is divisible by 2 and odd if and only if it is not.*

> Since x is prime and greater than 2, the definition of prime numbers means that x is divisible only by 1 and x. But if x is greater than 2, and is only divisible by 1 and itself, it is not divisible by 2. Thus, x is odd.

However, in class it wasn't clear how these definitions might be useful and so we considered another tactic, trying to show the contrapositive. Substituting symbols can help with getting this statement correct: let $p(x)$ be that x is prime, $g(x)$ be that $x > 2$, and $o(x)$ be that x is odd. Then the original statement is $p(x) \wedge g(x) \to o(x)$ and the contrapositive is $\neg o(x) \to \neg(p(x) \wedge g(x))$ or $\neg o(x) \to \neg p(x) \vee \neg g(x)$: if x is not odd (even), then x is either not prime or not greater than 2 (i.e. less than or equal to 2).

So then we try to proceed as before, utilizing the implication tactics:

> Assume that x is even.

And then making use of the relevant definitions —

> The definition of even means that x is thus divisible by 2. But a prime number is only divisible by 1 and itself, so if x is prime and divisible by 2, x must be 2 (which is even). Since x is either prime or not prime, x is either not prime or it is 2 (and thus not greater than 2).

This gives a slightly different route to the same conclusion.

While it isn't necessary to write out the proof formally in terms of predicate logic or to explicitly state the rules applied, those rules should still underlie the series of deductions you are making. Keeping in mind the symbolic form of what you are doing helps ensure that you are making valid deductions. Let's try that here.

Let $p(x)$ be "x is prime", $g(x, a)$ be "$x > a$", and $o(x)$ be "x is odd".

Let x be an arbitrary number. We want to show that if x is a prime number greater than 2, x is odd.

$$\frac{p(x) \land g(x, 2)}{\therefore o(x)}$$

Assume that x is a prime number greater than 2.

$p(x) \land g(x, 2)$ (premise)

An integer $n > 1$ is prime if it is divisible by exactly two positive numbers, 1 and itself.
An integer is odd if and only if it is not divisible by 2.

$\forall n((g(n, 1) \land p(n)) \rightarrow$
$(d(n, 1) \land d(n, n) \land \forall y((y \neq n \land y \neq 1) \rightarrow \neg d(n, y))))$
(definition of prime numbers)
$\forall n(o(n) \leftrightarrow \neg d(n, 2))$ (definition of odd numbers)

Since x is prime and greater than 2, the definition of prime numbers means that x is divisible only by 1 and x.

$g(x, 2)$ (premise, $p \land q \therefore p$)
$g(x, 2) \rightarrow g(x, 1)$ (math)
$g(x, 1)$ (*modus ponens*)

$(g(x, 1) \land p(x)) \rightarrow$
$(d(x, 1) \land d(x, n) \land \forall y((y \neq x \land y \neq 1) \rightarrow \neg d(x, y)))$
definitions, *modus ponens*)
$d(x, 1) \land d(x, n) \land \forall y((y \neq x \land y \neq 1) \rightarrow \neg d(x, y))$
(*modus ponens*)

But if x is greater than 2, and is only divisible by 1 and itself, it is not divisible by 2.

$d(x, 1) \land d(x, n) \land ((2 \neq x \land 2 \neq 1) \rightarrow \neg d(x, 2))$
(*modus ponens*)
$(2 \neq x) \rightarrow \neg d(x, 2)$ ($p \land \top \equiv p$, $p \land q \therefore p$)
$g(x, 2) \rightarrow (x \neq 2)$ (math)
$\neg d(x, 2)$ (law of syllogism, *modus ponens*)

	$o(x) \leftrightarrow \neg d(x, 2)$	(*modus ponens*)
Thus, x is odd.	$o(x)$	(definitions, *modus ponens*)