Logic and Proof

Key Points

- *logical deduction* is applying rules to a set of premises to generate conclusions
  - a mechanical process – logical deduction is a form of computation
- *propositional logic* deals with statements that have truth values (true/false)
- this is foundational
  - computers are built of circuits that implement operations on on/off (true/false) values
  - logical deduction provides a way to generate new knowledge without having to explicitly store every fact

Objectives

- objectives
  - learn the rules of logic and understand why they are valid
  - be able to apply the rules of logic in both symbolic form and English to generate new conclusions and to prove or disprove particular statements

Key Points – Propositional Logic

- definitions – proposition, propositional variable, mathematical generality
- logical operators
  - symbols and meaning
  - precedence rules
  - meaning of associativity
- truth tables
- terms and concepts – contrapositive, converse, tautology, logical equivalence, contradiction
- applying propositional logic to English statements

Propositional Logic

- a *proposition* is a statement that has a truth value (true or false)
- a *propositional variable* is a name that stands in for a specific proposition
  - conventionally a lowercase letter p, q, r
  - allows for *mathematical generality* – a statement about p holds for every possible proposition to p could represent
- a *compound proposition* combines simpler propositions and logical operators
  - conventionally represented by a capital letter P (Q, R)

Logical Operators

name	written as	read as	alternate symbol(s)		meaning
conjunction, and	p ∧ q	and		AND	*both p and q* true only when both p and q are true, otherwise false
disjunction, or	p ∨ q	or		OR	*either p or q, or both* true only when at least one of *p* and *q* are true false only when both *p* and *q* are false
negation, not	¬p	not	~	NOT	*the opposite of p* true when *p* is false, false when *p* is true
conditional	p → q	implies, or if *p* then *q*		IMPLIES	*both p and q are true, or p is false* false only when *p* is true and *q* is false, otherwise true
biconditional	p ↔ q	if and only if	≡, ⇔	IFF	*p implies q, and q implies p* true only when *p* and *q* have the same value (either both true or both false), false when *p* and *q* have different values
exclusive or, xor	p ⊕ q	xor		XOR	*either p or q, but not both* true only when *p* and *q* have different values (one is true and the other false), false when *p* and *q* have the same value (both true or both false)

Truth Tables

- a *truth table* expresses the value of a compound proposition for each combination of values for its propositional variables
- can be used to define operators or show *logical equivalence*

p	q	$p \rightarrow q$	$p \leftrightarrow q$	$p \oplus q$
false	false	true	true	false
false	true	true	false	true
true	false	false	false	true
true	true	true	true	false

p	q	r	$p \wedge q$	$q \wedge r$	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$
false	false	false	false	false	false	false
false	false	true	false	false	false	false
false	true	false	false	false	false	false
false	true	true	false	true	false	false
true	false	false	false	false	false	false
true	false	true	false	false	false	false
true	true	false	true	false	false	false
true	true	true	true	true	true	true

Precedence

- why precedence rules?
  - need an unambiguous interpretation of a compound proposition involving multiple operators
- from highest to lowest: ¬, ∧, ∨ and ⊕, →, ↔
- priority is left to right in a series of operators of equal precedence
- use parentheses to override precedence or add clarity

2. Insert parentheses into the following compound propositions to show the order in which the operators are evaluated:
a) $\neg p \vee q$ **b)** $p \wedge q \vee \neg p$ **c)** $p \vee q \wedge r$ **d)** $p \wedge \neg q \vee r$

Terms and Concepts

- *p* → *q* is an *implication*
- (¬q) → (¬p) is the *contrapositive* of p → q
  - an implication is logically equivalent to its contrapositive

  If this is Tuesday, then we are in Belgium.
  If we aren't in Belgium, then this isn't Tuesday.

- q → p is the *converse* of p → q
  - an implication is *not* logically equivalent to its converse

  If this is Tuesday, then we are in Belgium.
  If we are in Belgium, then this is Tuesday.

  maybe we spend both Tuesday and Wednesday in Belgium

Terms and Concepts

- *P* is a *tautology* if and only if it is true for all possible combinations of truth values for its propositional variables
- *P* is a *contradiction* if and only if it is false for all possible combinations of truth values for its propositional variables

4. Some of the following compound propositions are tautologies, some are contradictions, and some are neither. In each case, use a truth table to decide to which of these categories the proposition belongs:

a) $(p \wedge (p \to q)) \to q$ **b)** $((p \to q) \wedge (q \to r)) \to (p \to r)$
c) $p \wedge (\neg p)$ **d)** $(p \vee q) \to (p \wedge q)$
e) $p \vee (\neg p)$ **f)** $(p \wedge q) \to (p \vee q)$

Terms and Concepts

- *P* and *Q* are *logically equivalent* if and only if *P* ↔ *Q* is a tautology
  - written *P* ≡ *Q*

5. Use truth tables to show that each of the following propositions is logically equivalent to $p \leftrightarrow q$.

a) $(p \to q) \wedge (q \to p)$ **b)** $(\neg p) \leftrightarrow (\neg q)$
c) $(p \to q) \wedge ((\neg p) \to (\neg q))$ **d)** $\neg(p \oplus q)$

Reasoning with Propositional Logic

- a proposition is expressed in English as a sentence which says something (the *predicate*) about the sentence's *subject*
- a caution
  - English can express more and is more ambiguous than propositional logic

But English is a little too rich for mathematical logic. When you read the sentence "I wanted to leave and I left," you probably see a connotation of causality: I left *because* I wanted to leave. This implication does not follow from the logical combination of the truth values of the two propositions "I wanted to leave" and "I left." Or consider the proposition "I wanted to leave but I did not leave." Here, the word "but" has the same *logical* meaning as the word "and," but the connotation is very different. So, in

Implication and Deduction

- $p \rightarrow q$ is true only if both p and q are true, or if p is false

Pigs can't fly; airplanes can. Mice are small; the earth is not flat.

Which are valid statements? is $p \rightarrow q$ true?

If pigs can fly, then mice are small. ✓
If pigs can fly, then the earth is flat. ✓
it is valid to claim anything if p is false

If airplanes can fly, then mice are small. ✓
If airplanes can fly, then the earth is flat. ✘
it is not valid to claim a false thing if p is true

Implication and Deduction

- $p \rightarrow q$ is true in two situations – if both p and q are true, or if p is false

I say – If the party is on Tuesday, then I'll be there. $p \rightarrow q$ is true

The party is on Tuesday. Will I be there? p is true, is q?

✓ yes, because when p is true, $p \rightarrow q$ is only true if q also is

The party is on Friday. Will I be there? p is not true, is q?

who knows? $p \rightarrow q$ is always true when p is false, regardless of whether q is true

7. Let p represent the proposition "You leave" and let q represent the proposition "I leave." Express the following sentences as compound propositions using p and q, and show that they are logically equivalent:
a) Either you leave or I do. (Or both!)
b) If you don't leave, I will.

8. Suppose that m represents the proposition "The Earth moves," c represents "The Earth is the center of the universe," and g represents "Galileo was railroaded." Translate each of the following compound propositions into English:
a) $\neg g \wedge c$ **b)** $m \rightarrow \neg c$
c) $m \leftrightarrow \neg c$ **d)** $(m \rightarrow g) \wedge (c \rightarrow \neg g)$

9. Give the converse and the contrapositive of each of the following English sentences:
a) If you are good, Santa brings you toys.
b) If the package weighs more than one ounce, then you need extra postage.
c) If I have a choice, I don't eat eggplant.

Writing Up Solutions

- "writing up a solution" is a thing
  - a separate step from figuring out the solution
  - not necessarily or only a dump of the process you went through to find the solution
- give the solution *and* provide support for the solution
  - "support" provides evidence for the correctness of the solution
  - unsupported answers will receive little or no credit
- writeup should be organized, neat, and readable
  - write English in sentences, use paragraphs, etc
  - don't waste time recopying if there's a small mistake, but it should be more polished than an evolving draft
- typing is encouraged
  - encourages revision
  - OK to handwrite e.g. if typing isn't time-efficient for particular content

- use truth tables to prove logical equivalency

a) $p \vee (q \vee p) \equiv p \vee q$
b) $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

- convert to propositional logic

a) Jack is smart but not lucky.
b) If I have a choice, then I don't eat broccoli.
c) Achilles is brave and famous or long-lived.