Key Points

- the elements of boolean algebra
  - elements
  - operators
  - rules
- a common sense understanding of the rules
- applying the rules
- showing logical equivalence by finding chains of equivalences

The Elements of Boolean Algebra

- an algebra consists of a set of elements, operations defined on those elements, and a set of rules that govern the behavior of those operations
- boolean algebra
  - elements: true ($\mathbb{T}$), false ($\mathbb{F}$)
  - operators: ∧, ∨, ¬, →, ↔, ⊕, ≡
  - rules →

Double negation	$\neg(\neg p) \equiv p$
Excluded middle	$p \vee \neg p \equiv \mathbb{T}$
Contradiction	$p \wedge \neg p \equiv \mathbb{F}$
Identity laws	$\mathbb{T} \wedge p \equiv p$ $\mathbb{F} \vee p \equiv p$
Idempotent laws	$p \wedge p \equiv p$ $p \vee p \equiv p$
Commutative laws	$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$
Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
DeMorgan's laws	$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$ $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$

Understanding the Rules

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note: common sense and obviousness are not proofs – prove that these laws are true with truth tables

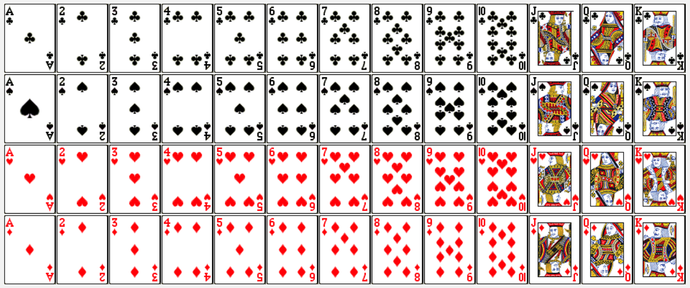
excluded middle: at least one of p or ¬p must be true
contradiction: it is not possible for both p and ¬p to be true

identity: always true regardless of the values of the variables
idempotent: an element whose value is unchanged when operated on by itself

distributive law:
ace ∧ (spades ∨ clubs) ≡
(ace ∧ spades) ∨ (ace ∧ clubs)
this card is an ace and either a spade or a club is equivalent to
this card is the ace of spades or the ace of clubs

DeMorgan's laws:
¬(queen ∧ spades) ≡ ¬queen ∨ ¬spade
this card is not the queen of spades
is equivalent to
this card is not a queen or it is not a spade (or both)

¬(queen ∨ spades) ≡ ¬queen ∧ ¬spade
it is not the case that this card is a queen or a spade
is equivalent to
this card is not a queen and not a spade



Rules Summary

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definitions –

$p \rightarrow q \equiv \neg p \vee q$

$p \leftrightarrow q \equiv (\neg p \vee q) \wedge (\neg q \vee p)$

- *duality* – for any tautology that uses only ∧ , ∨ , ¬, another tautology can be obtained by interchanging ∧ with ∨ , and $\mathbb{T}$ with $\mathbb{F}$
- *First Substitution Law* – for any tautology containing *p*, another tautology can be obtained by replacing all occurrences of *p* with (*Q*)
- *Second Substitution Law* – if P ≡ Q, substituting Q for any occurrence of P in R results in a logically equivalent proposition
- *chaining logical equivalences* – P ≡ R follows from P ≡ Q and Q ≡ R

9. For each of the following pairs of propositions, show that the two propositions are logically equivalent by finding a chain of equivalences from one to the other. State which definition or law of logic justifies each equivalence in the chain.

a) $p \wedge (q \wedge p), \quad p \wedge q$
b) $(\neg p) \rightarrow q, \quad p \vee q$
c) $(p \vee q) \wedge \neg q, \quad p \wedge \neg q$
d) $p \rightarrow (q \rightarrow r), \quad (p \wedge q) \rightarrow r$
e) $(p \rightarrow r) \wedge (q \rightarrow r), \quad (p \vee q) \rightarrow r$
f) $p \rightarrow (p \wedge q), \quad p \rightarrow q$

10. For each of the following compound propositions, find a simpler proposition that is logically equivalent. Try to find a proposition that is as simple as possible.

a) $(p \wedge q) \vee \neg q$
b) $\neg(p \vee q) \wedge p$
c) $p \rightarrow \neg p$
d) $\neg p \wedge (p \vee q)$
e) $(q \wedge p) \rightarrow q$
f) $(p \rightarrow q) \wedge (\neg p \rightarrow q)$

11. Express the negation of each of the following sentences in natural English:

a) It is sunny and cold.
b) I will have cake or I will have pie.
c) If today is Tuesday, this is Belgium.
d) If you pass the final exam, you pass the course.

12. Apply one of the laws of logic to each of the following sentences, and rewrite it as an equivalent sentence. State which law you are applying.

a) I will have coffee and cake or pie.
b) He has neither talent nor ambition.
c) You can have spam, or you can have spam.