Key Points

- terms and concepts – predicate, predicate logic, domain of discourse, *n*-place predicate, open statement, free variable, bound variable
- the elements of predicate logic
  - predicates
  - quantifiers
  - additional rules
- the notions of *tautology* and *logical equivalence* applied to predicate logic
- applying predicate logic to English statements

Predicate Logic

- so far, our propositions have been blanket statements that are true or false

Roses are red. Pigs can fly.

- predicate logic allows for true/false statements about particular entities

This rose is red. Wilbur can fly.
Some roses are red. All roses are red.
No pigs can fly.

  - the *predicate* is the property part

is red
can fly

Predicates and Propositions

- *P(a)* stands for the proposition formed when predicate *P* is applied to entity *a*
  - *a* must belong to the *domain of discourse* for P
    - similar to the notion of parameter types for functions in Java
- an *n*-place predicate involves *n* entities

Wilbur can fly – fly(p)
Alice is friends with *Bob* – friends(a,b)
Alice gave *Bob a book* – gave(a,b,obj)
Alice bought *a plant* from *Bob* for *$10* – bought(a,obj,b,price)

- *P(a)* is a proposition, and the rules of propositional logic apply

friends(a,b) ↔ friends(b,a)

Quantifiers

- *quantifiers* allow the application of predicates to more than individual entities
- the *universal quantifier* ∀ expresses “for all” or “every”
  - $\forall x P(x)$ is true if and only if $P(a)$ is true for every entity a in the domain of discourse

All roses are red.
____ is red is true for every rose.

- the *existential quantifier* ∃ expresses “there exists”, “some”, “for at least one”
  - $\exists x P(x)$ is true if and only if there’s at least one entity a in the domain of discourse for which $P(a)$ is true

Some roses are red.
____ is red is true for at least one rose.

6. Let $T(x, y)$ stand for "x has taken y," where the domain of discourse for x consists of students and the domain of discourse for y consists of math courses (at your school). Translate each of the following propositions into an unambiguous English sentence:

a) $\forall x\, \forall y\, T(x, y)$ **b)** $\forall x\, \exists y\, T(x, y)$ **c)** $\forall y\, \exists x\, T(x, y)$
d) $\exists x\, \exists y\, T(x, y)$ **e)** $\exists x\, \forall y\, T(x, y)$ **f)** $\exists y\, \forall x\, T(x, y)$

8. Translate each of the following sentences into a proposition using predicate logic. Make up any predicates you need. State what each predicate means and what its domain of discourse is.

a) All crows are black.
b) Any white bird is not a crow.
c) Not all politicians are honest.
d) All green elephants have purple feet.
e) There is no one who does not like pizza.
f) Anyone who passes the final exam will pass the course.
g) If x is any positive number, then there is a number y such that $y^2 = x$.

Terms and Concepts

- $\forall x P(x)$, $\exists x P(x)$, and $P(a)$ are propositions
  - x is an entity variable, a is an entity

these are statements which can be true or false

- $P(x)$ is not a proposition
  - x is an entity variable

x is just a placeholder – e.g. ___ is red isn't true or false

⚠ a convention is x, y, z for variables and a, b, c for entities, but any letter can be used for either concept – must pay attention to context to know what's up

- an *open statement* contains one or more entity variables
  - must fill in those "slots" with particular entities to get a proposition
- an entity variable is *bound* if quantified with $\forall x$ or $\exists x$ and *free* otherwise
  - an *open statement* contains one or more free variables
  - an expression with no free variables is a proposition

Rules of Predicate Logic

Double negation	$\neg(\neg p) \equiv p$
Excluded middle	$p \vee \neg p \equiv \mathbb{T}$
Contradiction	$p \wedge \neg p \equiv \mathbb{F}$
Identity laws	$\mathbb{T} \wedge p \equiv p$
	$\mathbb{F} \vee p \equiv p$
Idempotent laws	$p \wedge p \equiv p$
	$p \vee p \equiv p$
Commutative laws	$p \wedge q \equiv q \wedge p$
	$p \vee q \equiv q \vee p$
Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
DeMorgan's laws	$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$
	$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$

$\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$
$\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$
$\forall x \forall y Q(x, y) \equiv \forall y \forall x Q(x, y)$
$\exists x \exists y Q(x, y) \equiv \exists y \exists x Q(x, y)$

DeMorgan's laws for predicate logic

$p \rightarrow q \equiv \neg p \vee q$

$p \leftrightarrow q \equiv (\neg p \vee q) \wedge (\neg q \vee p)$

Let $\mathcal{P}$ and $\mathcal{Q}$ be predicate logic formulas containing predicate variables.

$\mathcal{P}$ is a *tautology* if is it true whenever all of its predicate variables are replaced by actual predicates

$\mathcal{P}$ is *logically equivalent* to $\mathcal{Q}$ (written $\mathcal{P} \equiv \mathcal{Q}$) if $\mathcal{P} \leftrightarrow \mathcal{Q}$ is a tautology

1. Simplify each of the following propositions. In your answer, the $\neg$ operator should be applied only to individual predicates.

a) $\neg \forall x(\neg P(x))$ **b)** $\neg \exists x(P(x) \wedge Q(x))$
c) $\neg \forall z(P(z) \rightarrow Q(z))$ **d)** $\neg((\forall x P(x)) \wedge \forall y(Q(y)))$
e) $\neg \forall x \exists y P(x, y)$ **f)** $\neg \exists x(R(x) \wedge \forall y S(x, y))$
g) $\neg \exists y(P(y) \leftrightarrow Q(y))$ **h)** $\neg(\forall x(P(x) \rightarrow (\exists y Q(x, y))))$

3. Find the negation of each of the following propositions. Simplify the result; in your answer, the $\neg$ operator should be applied only to individual predicates.

a) $\neg \exists n(\forall s C(s, n))$
b) $\neg \exists n(\forall s(L(s, n) \rightarrow P(s)))$
c) $\neg \exists n(\forall s(L(s, n) \rightarrow (\exists x \exists y \exists z Q(x, y, z))))$.
d) $\neg \exists n(\forall s(L(s, n) \rightarrow (\exists x \exists y \exists z(s = xyz \wedge R(x, y) \wedge T(y) \wedge U(x, y, z))))$.