Proof Strategies

Show p → q.

- for a *direct proof*, show p→q
  - assume p, deduce q

- for an *indirect proof*, show the contrapositive ¬q→¬p
  - assume ¬q, deduce ¬p
  - conclude that p→q is true because of the logical equivalence of p→q and ¬q→¬p

- *proof by contradiction*
  - assume the opposite of what you trying to show (i.e. p∧¬q), deduce a contradiction
    - p→q ≡ ¬p ∨ q means that ¬(p→q) ≡ p∧¬q
  - conclude that p→q is true because the only flaw was assuming ¬q
    - if we start with a set of assumptions and apply rules of logic to derive a statement that is known to be false, then at least one of the assumptions must be false

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- The **natural numbers** (denoted $\mathbb{N}$) are the numbers $0, 1, 2, \ldots$. Note that the sum and product of natural numbers are natural numbers.

- The **integers** (denoted $\mathbb{Z}$) are the numbers $0, -1, 1, -2, 2, -3, 3, \ldots$. Note that the sum, product, and difference of integers are integers.

- The **rational numbers** (denoted $\mathbb{Q}$) are all numbers that can be written in the form $\frac{m}{n}$ where m and n are integers and $n \neq 0$. So $\frac{1}{3}$ and $\frac{-65}{7}$ are rationals; so, less obviously, are 6 and $\frac{\sqrt{27}}{\sqrt{12}}$ since $6 = \frac{6}{1}$ (or, for that matter, $6 = \frac{-12}{-2}$), and $\frac{\sqrt{27}}{\sqrt{12}} = \sqrt{\frac{27}{12}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$. Note the restriction that the number in the denominator cannot be 0: $\frac{3}{0}$ is not a number at all, rational or otherwise; it is an undefined quantity. Note also that the sum, product, difference, and quotient of rational numbers are rational numbers (provided you don't attempt to divide by 0.)

- The **real numbers** (denoted $\mathbb{R}$) are numbers that can be written in decimal form, possibly with an infinite number of digits after the decimal point. Note that the sum, product, difference, and quotient of real numbers are real numbers (provided you don't attempt to divide by 0.)

- The **irrational numbers** are real numbers that are not rational, i.e. that cannot be written as a ratio of integers. Such numbers include $\sqrt{3}$ (which we will prove is not rational) and π (if anyone ever told you that $\pi = \frac{22}{7}$, they lied—$\frac{22}{7}$ is only an *approximation* of the value of π).

- An integer n is **divisible by** m iff $n = mk$ for some integer k. (This can also be expressed by saying that m **e**venly divides n.) So for example, n is divisible by 2 iff $n = 2k$ for some integer k; n is divisible by 3 iff $n = 3k$ for some integer k, and so on. Note that if n is *not* divisible by 2, then n must be 1 more than a multiple of 2 so $n = 2k+1$ for some integer k. Similarly, if n is not divisible by 3 then n must be 1 or 2 more than a multiple of 3, so $n = 2k + 1$ or $n = 2k + 2$ for some integer k.

- An integer is **even** iff it is divisible by 2 and **odd** iff it is not.

- An integer $n > 1$ is **prime** if it is divisible by exactly two positive integers, namely 1 and itself. Note that a number must be greater than 1 to even have a chance of being termed "prime". In particular, neither 0 nor 1 is prime.

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1. Suppose that a_1, a_2, ..., a_{10} are real numbers, and suppose that $a_1 + a_2 + \cdots + a_{10} > 100$. Use a proof by contradiction to conclude that at least one of the numbers a_i must be greater than 10.

3. The **pigeonhole principle** is the following obvious observation: If you have n pigeons in k pigeonholes and if $n > k$, then there is at least one pigeonhole that contains more than one pigeon. Even though this observation seems obvious, it's a good idea to prove it. Prove the pigeonhole principle using a proof by contradiction.

2. Prove that each of the following statements is true. In each case, use a proof by contradiction. Remember that the negation of $p \to q$ is $p \land \neg q$.
  - **a)** Let n be an integer. If n^2 is an even integer, then n is an even integer.
  - **b)** $\sqrt{2}$ is irrational.
  - **c)** If r is a rational number and x is an irrational number, then $r + x$ is an irrational number. (That is, the sum of a rational number and an irrational number is irrational.)
  - **d)** If r is a non-zero rational number and x is an irrational number, then rx is an irrational number.
  - **e)** If r and $r + x$ are both rational, then x is rational.

4. Determine whether each of the following statements is true. If it true, prove it. If it is false, give a counterexample.

    a) Every prime number is odd.

    b) Every prime number greater than 2 is odd.

    c) If x and y are integers with $x < y$, then there is an integer z such that $x < z < y$.

    d) If x and y are real numbers with $x < y$, then there is a real number z such that $x < z < y$.

5. Suppose that r, s, and t are integers, such that r evenly divides s and s evenly divides t. Prove that r evenly divides t.

6. Prove that for all integers n, if n is odd then n^2 is odd.

7. Prove that an integer n is divisible by 3 iff n^2 is divisible by 3. (Hint: give an indirect proof of "if n^2 is divisible by 3 then n is divisible by 3.")

8. Prove or disprove each of the following statements.

    a) The product of two even integers is even.

    b) The product of two integers is even only if both integers are even.

    c) The product of two rational numbers is rational.

    d) The product of two irrational numbers is irrational.

    e) For all integers n, if n is divisible by 4 then n^2 is divisible by 4.

    f) For all integers n, if n^2 is divisible by 4 then n is divisible by 4.