Proof by Induction

We can prove a statement *P(x)* for any natural number *x* by showing a chain of statements

P(0), P(0) → P(1), P(1) → P(2), P(2) → P(3), …

Thus, prove

P(0) ∧ ∀k (P(k) → P(k+1))

- *proof by induction*
  - base case: show P(0)
  - inductive case: show ∀k (P(k) → P(k+1))
    - tactic: let *k* be an arbitrary element of ℕ and prove *P(k) → P(k+1)*
      - P(k) is the *inductive hypothesis*
    - another tactic: show (P(0) ∧ P(1) ∧ P(2) ∧ … ∧ P(k)) → P(k+1)
      - assume *P(x)* holds for all *x* 0 to *k*, then show P(k+1)
      - known as *strong induction*

1. Use induction to prove that $n^3 + 3n^2 + 2n$ is divisible by 3 for all natural numbers n.

2. Use induction to prove that

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$$

for any natural number n and for any real number r such that $r \neq 1$.

3. Use induction to prove that for any natural number n,

$$\sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n}$$

4. Use induction to prove that for any natural number n,

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

5. Use induction to prove that for any positive integer n,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

6. Use induction to prove that for any positive integer n,

$$\sum_{i=1}^{n} (2i - 1) = n^2$$

10. Use induction to prove the following generalized distributive laws for propositional logic: For any natural number $n > 1$ and any propositions q, p_1, p_2, …, p_n,

a) $q \wedge (p_1 \vee p_2 \vee \cdots \vee p_n) = (q \wedge p_1) \vee (q \wedge p_2) \vee \cdots \vee (q \wedge p_n)$

b) $q \vee (p_1 \wedge p_2 \wedge \cdots \wedge p_n) = (q \vee p_1) \wedge (q \vee p_2) \wedge \cdots \wedge (q \vee p_n)$

- The ***natural numbers*** (denoted $\mathbb{N}$) are the numbers $0, 1, 2, \ldots$. Note that the sum and product of natural numbers are natural numbers.
- The ***integers*** (denoted $\mathbb{Z}$) are the numbers $0, -1, 1, -2, 2, -3, 3, \ldots$. Note that the sum, product, and difference of integers are integers.
- The ***rational numbers*** (denoted $\mathbb{Q}$) are all numbers that can be written in the form $\frac{m}{n}$ where m and n are integers and $n \neq 0$. So $\frac{1}{3}$ and $\frac{-65}{7}$ are rationals; so, less obviously, are 6 and $\frac{\sqrt{27}}{\sqrt{12}}$ since $6 = \frac{6}{1}$ (or, for that matter, $6 = \frac{-12}{-2}$), and $\frac{\sqrt{27}}{\sqrt{12}} = \sqrt{\frac{27}{12}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$. Note the restriction that the number in the denominator cannot be 0: $\frac{3}{0}$ is not a number at all, rational or otherwise; it is an undefined quantity. Note also that the sum, product, difference, and quotient of rational numbers are rational numbers (provided you don't attempt to divide by 0.)
- The ***real numbers*** (denoted $\mathbb{R}$) are numbers that can be written in decimal form, possibly with an infinite number of digits after the decimal point. Note that the sum, product, difference, and quotient of real numbers are real numbers (provided you don't attempt to divide by 0.)
- The ***irrational numbers*** are real numbers that are not rational, i.e. that cannot be written as a ratio of integers. Such numbers include $\sqrt{3}$ (which we will prove is not rational) and π (if anyone ever told you that $\pi = \frac{22}{7}$, they lied—$\frac{22}{7}$ is only an *approximation* of the value of π).

- An integer n is ***divisible by*** m iff $n = mk$ for some integer k. (This can also be expressed by saying that m ***e***venly divides n.) So for example, n is divisible by 2 iff $n = 2k$ for some integer k; n is divisible by 3 iff $n = 3k$ for some integer k, and so on. Note that if n is *not* divisible by 2, then n must be 1 more than a multiple of 2 so $n = 2k+1$ for some integer k. Similarly, if n is not divisible by 3 then n must be 1 or 2 more than a multiple of 3, so $n = 2k + 1$ or $n = 2k + 2$ for some integer k.

- An integer is ***even*** iff it is divisible by 2 and ***odd*** iff it is not.
- An integer $n > 1$ is ***prime*** if it is divisible by exactly two positive integers, namely 1 and itself. Note that a number must be greater than 1 to even have a chance of being termed "prime". In particular, neither 0 nor 1 is prime.