Functions

- a *functional relationship* between sets *A* and *B* associates exactly one element of *B* with each element of *A*
- also called a *mapping* from set *A* to set *B*
- a *function* expresses a functional relationship
  - written $f: A \to B$
    - *f* is a function from *A* to *B*
    - *f* maps *A* to *B*
  - for $a \in A$, $f(a) \in B$ is the element of *B* that *f* associates with *a*
    - *f*(*a*) is the *value* of *f* at *a*
  - note that while there is exactly one element of *B* associated with a given *a*, it is not required that it be unique i.e. it is not required that $f(a_1) \neq f(a_2)$ for $a_1 \neq a_2$
- $g \circ f$ is the *composition* of *g* and *f*: $(g \circ f)(a) = g(f(a))$
  - to be valid, requires that $f: A \to B$ and $g: B \to C$

Ordered *n*-tuples

- (*a*,*b*) is the *ordered pair* containing entities *a* and *b*
  - if $a \neq b$, (*a*,*b*) and (*b*,*a*) are different
  - (*a*,*b*) and (*c*,*d*) are equal iff $a = c$ and $b = d$
- for sets *A* and *B*, $A \times B$ is the *cross product* (or *Cartesian product*)

$$A \times B = \{(a,b) | a \in A \wedge b \in B\}$$

  - contains every ordered pair containing an element of *A* and an element of *B*
  - extends to *ordered triples* $A \times B \times C$ and other *ordered n-tuples*
- the set $\{ (a,b) \in A \times B \mid a \in A \text{ and } b = f(a) \}$ is the *graph* of *f*
  - a function can be specified by giving a set of ordered pairs
- *f*((*a*,*b*)) is more commonly written *f*(*a*,*b*)

1. Let $A = \{1, 2, 3, 4\}$ and let $B = \{a, b, c\}$. Find the sets $A \times B$ and $B \times A$.

2. Let A be the set $\{a, b, c, d\}$. Let f be the function from A to A given by the set of ordered pairs $\{(a, b), (b, b), (c, a), (d, c)\}$, and let g be the function given by the set of ordered pairs $\{(a, b), (b, c), (c, d), (d, d)\}$. Find the set of ordered pairs for the composition $g \circ f$.

3. Let $A = \{a, b, c\}$ and let $B = \{0, 1\}$. Find all possible functions from A to B. Give each function as a set of ordered pairs. (Hint: Every such function corresponds to one of the subsets of A.)

More Terminology

for a function $f: A \to B$

- *A* is the *domain* of *f*
- *B* is the *range* of *f*
- the *image* of *f* is $\{ f(a) \mid a \in A \}$
  - in some contexts, this is known as the range

the image is the subset of the range containing the values of *f*

- function *f* is *onto* (or *surjective*) if the image is equal to the range $\forall b \in B (\exists a \in A (b = f(a)))$
- function *f* is *one-to-one* (or *injective*) if each element of the range is associated with at most one element of the domain $\forall x \in A \forall y \in A (x \neq y \rightarrow f(x) \neq f(y))$ $\forall x \in A \forall y \in A (f(x) = f(y) \rightarrow x = y)$
- function *f* is *bijective* if it is both one-to-one and onto

4. Consider the functions from $\mathbb{Z}$ to $\mathbb{Z}$ which are defined by the following formulas. Decide whether each function is onto and whether it is one-to-one; prove your answers.

a) $f(n) = 2n$ **b)** $g(n) = n + 1$ **c)** $h(n) = n^2 + n + 1$

d) $s(n) = \begin{cases} \text{n/2}, & \text{if } n \text{ is even} \\ \text{(n+1)/2}, & \text{if } n \text{ is odd} \end{cases}$