One-to-One Correspondence

- a *one-to-one correspondence* between sets *A* and *B* means every element of *A* is paired with an element of *B* and every element of *B* is paired with an element of *A*
  - demonstrates that two sets have the same number of elements
  - counting establishes a one-to-one correspondence between a set with *n* elements and the set of numbers { 1, 2, …, *n* }

Theorem 2.6. *For each $n \in \mathbb{N}$, let N_n be the set $N_n = \{0, 1, \ldots, n-1\}$. If $n \neq m$, then there is no bijective function from N_m to N_n.*

  - *bijective* is both one-to-one and onto
    - *one-to-one* – each element of the range is associated with at most one element of the domain
    - *onto* – the image equals the range
  - there can't be a one-to-one correspondence between sets of a different size

Finite and Infinite

- a set *A* is *finite* is there is a one-to-one correspondence between *A* and $\mathbb{N}_n$ for some natural number *n*
- a set *A* is *infinite* if there is no such *n*

Cardinality

- for a finite set *A*, the *cardinality* of *A*, written |*A*|, is the number of elements in *A*

Theorem 2.7. *A finite set with cardinality n has 2^n subsets.*

Theorem 2.8. *Let A and B be finite sets. Then*

- $|A \times B| = |A| \cdot |B|$.
- $|A \cup B| = |A| + |B| - |A \cap B|$.
- *If A and B are disjoint then* $|A \cup B| = |A| + |B|$.
- $|A^B| = |A|^{|B|}$.
- $|\mathcal{P}(A)| = 2^{|A|}$.

A^B is the set of functions $B \to A$

correction! in class A^B was incorrectly defined as the set of functions A → B – it should instead be the set of functions B → A, making $|A^B| = |A|^{|B|}$ as stated in the theorem

1. Suppose that *A*, *B*, and *C* are finite sets which are pairwise disjoint. (That is, $A \cap B = A \cap C = B \cap C = \emptyset$.) Express the cardinality of each of the following sets in terms of $|A|$, $|B|$, and $|C|$. Which of your answers depend on the fact that the sets are pairwise disjoint?

a) $\mathcal{P}(A \cup B)$ **b)** $A \times (B^C)$ **c)** $\mathcal{P}(A) \times \mathcal{P}(C)$
d) $A^{B \times C}$ **e)** $(A \times B)^C$ **f)** $\mathcal{P}(A^B)$
g) $(A \cup B)^C$ **h)** $(A \cup B) \times A$ **i)** $A \times A \times B \times B$

2. Suppose that *A* and *B* are finite sets which are not necessarily disjoint. What are all the possible values for $|A \cup B|$?

3. Let's say that an "identifier" consists of one or two characters. The fist character is one of the twenty-six letters (A, B, . . . , C). The second character, if there is one, is either a letter or one of the ten digits (0, 1, . . . , 9). How many different identifiers are there? Explain your answer in terms of unions and cross products.

4. Suppose that there are five books that you might bring along to read on your vacation. In how many different ways can you decide which books to bring, assuming that you want to bring at least one? Why?

Infinities

- a set A is *countably infinite* if there is a one-to-one correspondence between $\mathbb{N}$ and A
- a set A is *countable* if it is either finite or countably infinite
  - it is possible in principle to make a list of all of the elements of A even if the list will go on forever
- a set A is *uncountable* otherwise
  - it is impossible to make a list of the elements of A

Infinities

- $\mathbb{Z}$ (integer) is countably infinite

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0  1 -1  2 -2  3 -3  ...
0  1  2  3  4  5  6  ...
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- $\mathbb{Q}$ (rationals) is countably infinite

$$\left(\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \frac{1}{5}, \frac{5}{1}, \frac{1}{6}, \frac{2}{5}, \ldots\right)$$

n+m = 1 n+m = 3 n+m = 4 n+m = 5 n+m = 6 n+m = 7

n+m = 2

10. Show that the set $\mathbb{N} \times \mathbb{N}$ is countable.

- use a diagonalization argument

$$\begin{matrix} (0,0) & (0,1) & (0,2) & \cdots \\ (1,0) & (1,1) & (1,2) & \cdots \\ (2,0) & (2,1) & (2,2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

$$(0,0), (0,1), (1,0), (0,2), (1,1), (2,0), (0,3), (1,2), (2,1), (3,0), \ldots$$

a) Suppose that A and B are countably infinite sets. Show that $A \cup B$ is countably infinite.

Infinities

- $\mathbb{R}$ (reals) is not countably infinite

0.**9**039893724987956129792765485794 5...
0.1**2**3493420940598759802392308345 49...
0.22**4**000432984362347093232799895 79...
0.500**0**000000000000000000000000000...
0.7774**3**449234234876990120909480009...
0.77755**5**555888888894988889800001 11...
0.123456**7**888888888888888880000000 0...
0.3483544**0**009848712712123940320577...
0.93473244**4**479004983409999909489 00...

for any such list, construct a new number by picking a number other than the bold one for each column

e.g. 0.813724613…

since the value picked for the *n*th column is always different from the *n*th column of the *n*th number in the list, it has at least one digit different from every number in the list – and is thus not itself in the list

- $\mathbb{R} \backslash \mathbb{Q}$ (irrationals) is not countably infinite

Theorem 2.9. *Suppose that X is an uncountable set, and that K is a countable subset of X. Then the set $X \setminus K$ is uncountable.*

Infinities

Theorem 2.11. *Let X be any set. Then there is no one-to-one correspondence between X and $\mathcal{P}(X)$.*

  – for finite sets, $|\mathcal{P}(X)| = 2^{|X|} > |X|$
  – the “larger” relationship holds for infinite sets too

- can construct an infinite series of increasingly larger infinities with $\mathbb{R}$, $\mathcal{P}(\mathbb{R})$, $\mathcal{P}(\mathcal{P}(\mathbb{R}))$, $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{R})))$, …