Infinities

- ℝ (reals) is not countably infinite

$0.\mathbf{9}0398937249879561297927654857945\ldots$
$0.1\mathbf{2}349342094059875980239230834549\ldots$
$0.22\mathbf{4}00043298436234709323279989579\ldots$
$0.500\mathbf{0}0000000000000000000000000000\ldots$
$0.7774\mathbf{3}449234234876990120909480009\ldots$
$0.77755\mathbf{5}55588888888949888898000111\ldots$
$0.123456\mathbf{7}8888888888888888800000000\ldots$
$0.3483544\mathbf{0}009848712712123940320577\ldots$
$0.93473244\mathbf{4}47900498340999990948900\ldots$

for any such list, construct a new number by picking a number other than the bold one for each column

e.g. 0.813724613…

since the value picked for the *n*th column is always different from the *n*th column of the *n*th number in the list, it has at least one digit different from every number in the list – and is thus not itself in the list

- ℝ\ℚ (irrationals) is not countably infinite

Theorem 2.9. *Suppose that X is an uncountable set, and that K is a countable subset of X. Then the set $X \setminus K$ is uncountable.*

Infinities

Theorem 2.11. *Let X be any set. Then there is no one-to-one correspondence between X and $\mathcal{P}(X)$.*

- for finite sets, $|\mathcal{P}(X)| = 2^{|X|} > |X|$
- the "larger" relationship holds for infinite sets too

- can construct an infinite series of increasingly larger infinities with ℝ, $\mathcal{P}(\mathbb{R})$, $\mathcal{P}(\mathcal{P}(\mathbb{R}))$, $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{R})))$, …

Relations

- there are many possible relationships between elements of sets
- a *functional relationship* between sets *A* and *B* associates exactly one element of *B* with each element of *A*
  - a function $f: A \rightarrow B$ can be thought of as a subset of $A \times B$ with certain properties
    - $\{ (a,b) \in A \times B \mid a \in A, b \in B, \text{ and } b = f(a) \}$
- relationships in general are captured by the notion of a *relation*
  - a *relation* $\mathcal{R}$ on *n* sets $A_1, A_2, \ldots, A_n$ is a subset of $A1 \times A_2 \times \ldots \times A_n$
  - for two sets A and B: $\mathcal{R} = \{ (a,b) \in A \times B \mid a \in A, b \in B \}$
  - for three sets A, B, C: $\mathcal{R} = \{ (a,b,c) \in A \times B \times C \mid a \in A, b \in B, c \in C \}$

Relations

- a *binary relation* $\mathcal{R}$ on *A* is a subset of $A \times A$
  - for a binary relation $\mathcal{R}$, $(a,b) \in \mathcal{R}$ can also be written as $a \mathcal{R} b$
  - examples
    - $n \leq m$
    - { (*c*,*p*) | *c* is a child of *p* }
- a *ternary relation* $\mathcal{R}$ on *A* is a subset of $A \times A \times A$
  - examples
    - $x^2+y^2+z^3 = 1$
- a *n-ary relation* $\mathcal{R}$ on *A* is a subset of $A \times A \times \ldots \times A$ with *n* copies of *A*

Properties of Binary Relations

- $\mathcal{R}$ is *reflexive* if $\forall a \in A\ (a\ \mathcal{R}\ a)$
  - every element in the set is related to itself
  - $n \le n$ is reflexive
  - $n < n$ is not

a) $\mathcal{R} = \{(a, b), (a, c), (a, d)\}$.
b) $\mathcal{S} = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a)\}$.
c) $\mathcal{T} = \{(b, b), (c, c), (d, d)\}$.
d) $\mathcal{C} = \{(a, b), (b, c), (a, c), (d, d)\}$.
e) $\mathcal{D} = \{(a, b), (b, a), (c, d), (d, c)\}$.

Properties of Binary Relations

- $\mathcal{R}$ is *transitive* if
  $\forall a \in A, \forall b \in A, \forall c \in A\ ((a\ \mathcal{R}\ b \wedge b\ \mathcal{R}\ c) \rightarrow (a\ \mathcal{R}\ c))$
  - *a* related to *b* and *b* related to *c* means that *a* is related to *c*
  - “child of” is not transitive – Alice is a child of Bob and Bob is a child of Carol but Alice is not a child of Carol
  - “descendant of” is transitive

a) $\mathcal{R} = \{(a, b), (a, c), (a, d)\}$.
b) $\mathcal{S} = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a)\}$.
c) $\mathcal{T} = \{(b, b), (c, c), (d, d)\}$.
d) $\mathcal{C} = \{(a, b), (b, c), (a, c), (d, d)\}$.
e) $\mathcal{D} = \{(a, b), (b, a), (c, d), (d, c)\}$.

Properties of Binary Relations

note typo in book: $b \in A$, not $b \in B$

- $\mathcal{R}$ is *symmetric* if $\forall a \in A, \forall b \in A\ (a\ \mathcal{R}\ b \rightarrow b\ \mathcal{R}\ a)$
  - whenever *a* is related to *b*, *b* is related to *a*
  - “sibling of” is symmetric – if Alice is Bob’s sibling, Bob is also Alice’s sibling
- $\mathcal{R}$ is *antisymmetric* if
  $\forall a \in A, \forall b \in A\ ((a\ \mathcal{R}\ b \wedge b\ \mathcal{R}\ a) \rightarrow a = b)$
  - we can’t have both that *a* is related to *b* and *b* is related to *a* for distinct elements *a*, *b*
  - $\le$ is antisymmetric – if $n \le m$ and $m \le n$, then $n = m$

a) $\mathcal{R} = \{(a, b), (a, c), (a, d)\}$.
b) $\mathcal{S} = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a)\}$.
c) $\mathcal{T} = \{(b, b), (c, c), (d, d)\}$.
d) $\mathcal{C} = \{(a, b), (b, c), (a, c), (d, d)\}$.
e) $\mathcal{D} = \{(a, b), (b, a), (c, d), (d, c)\}$.

- “not symmetric” is not the same as “antisymmetric”

4. It is possible for a relation to be both symmetric and antisymmetric. For example, the equality relation, $=$, is a relation on any set which is both symmetric and antisymmetric. Suppose that A is a set and $\mathcal{R}$ is a relation on A that is both symmetric and antisymmetric. Show that $\mathcal{R}$ is a subset of $=$ (when both relations are considered as sets of ordered pairs). That is, show that for any a and b in A, $(a\,\mathcal{R}\,b) \rightarrow (a = b)$.

- can you find an example of a relation that is neither symmetric nor antisymmetric?

Types of Binary Relations

- $\mathcal{R}$ is a *partial order* if –
  - it is reflexive,
  - it is antisymmetric, and
  - it is transitive
  - $\subseteq$ is a partial order on $\mathcal{P}(X)$
- $\mathcal{R}$ is a *total order* if –
  - it is a partial order, and
  - $\forall a \in A, \forall b \in A\ (a\ \mathcal{R}\ b \lor b\ \mathcal{R}\ a)$
  - $\subseteq$ is a *not* a total order on $\mathcal{P}(X)$
  - $\leq$ is a total order on $\mathbb{N}$

a) $\mathcal{R} = \{(a,b),\ (a,c),\ (a,d)\}$.
b) $\mathcal{S} = \{(a,a),\ (b,b),\ (c,c),\ (d,d),\ (a,b),\ (b,a)\}$.
c) $\mathcal{T} = \{(b,b),\ (c,c),\ (d,d)\}$.
d) $\mathcal{C} = \{(a,b),\ (b,c),\ (a,c),\ (d,d)\}$.
e) $\mathcal{D} = \{(a,b),\ (b,a),\ (c,d),\ (d,c)\}$.

Additional Definitions

- a *partition* of *A* is a collection of non-empty subsets of *A* such that –
  - each pair of distinct subsets is disjoint, and
  - the union of all subsets in the collection is *A*
  - i.e. a partition divides up all of the elements of *A* into separate sets

Additional Definitions

- an *equivalence relation* is a binary relation that is reflexive, symmetric, and transitive
- the *equivalence class* of *a* in *A* under equivalence relation $\mathcal{R}$, written $[a]_{\mathcal{R}}$, is $\{ b \in A \mid b\ \mathcal{R}\ a \}$
  - the equivalence class of *a* is the set of all elements that are related to *a* under an equivalence relation
  - the notation can be shortened to $[a]$ if $\mathcal{R}$ is understood

Theorem 2.12. *Let A be a set and let $\mathcal{R}$ be an equivalence relation on A. Then the collection of all equivalence classes under $\mathcal{R}$ is a partition of A.*

  - we can use equivalence relations to classify things into categories where the categories are equivalence classes

1. For a finite set, it is possible to define a binary relation on the set by listing the elements of the relation, considered as a set of ordered pairs. Let A be the set $\{a,b,c,d\}$, where a, b, c, and d are distinct. Consider each of the following binary relations on A. Is the relation reflexive? Symmetric? Antisymmetric? Transitive? Is it a partial order? An equivalence relation?
a) $\mathcal{R} = \{(a,b),\ (a,c),\ (a,d)\}$.
b) $\mathcal{S} = \{(a,a),\ (b,b),\ (c,c),\ (d,d),\ (a,b),\ (b,a)\}$.
c) $\mathcal{T} = \{(b,b),\ (c,c),\ (d,d)\}$.
d) $\mathcal{C} = \{(a,b),\ (b,c),\ (a,c),\ (d,d)\}$.
e) $\mathcal{D} = \{(a,b),\ (b,a),\ (c,d),\ (d,c)\}$.

2. Let A be the set $\{1,2,3,4,5,6\}$. Consider the partition of A into the subsets $\{1,4,5\}$, $\{3\}$, and $\{2,6\}$. Write out the associated equivalence relation on A as a set of ordered pairs.

3. Consider each of the following relations on the set of people. Is the relation reflexive? Symmetric? Transitive? Is it an equivalence relation?
a) x is related to y if x and y have the same biological parents.
b) x is related to y if x and y have at least one biological parent in common.
c) x is related to y if x and y were born in the same year.
d) x is related to y if x is taller than y.
e) x is related to y if x and y have both visited Honolulu.

Additional Definitions

- the *transitive closure* $\mathcal{R}^*$ of $\mathcal{R}$ captures the set of things related to each other by one or more steps
  - for $a, b \in A$, $a \,\mathcal{R}^*\, b$ if there is a sequence $a = x_0, x_1, \ldots, x_n = b$ for $x_i \in A$ and $n > 0$ such that $x_0 \,\mathcal{R}\, x_1, x_1 \,\mathcal{R}\, x_2, \ldots, x_{n-1} \,\mathcal{R}\, x_n$
  - let C be the set of all cities and $\mathcal{A}$ be the binary relation on C such that for $x, y \in C$, $x \,\mathcal{A}\, y$ if there is a regularly scheduled airline flight from x to y
    - what is $x \,\mathcal{A}^*\, y$?

10. Let P be the set of people and let $\mathcal{C}$ be the "child of" relation. That is $x \,\mathcal{C}\, y$ means that x is a child of y. What is the meaning of the transitive closure $\mathcal{C}^*$? Explain your answer.

11. Let $\mathcal{R}$ be the binary relation on $\mathbb{N}$ such that $x \,\mathcal{R}\, y$ if and only if $y = x + 1$. Identify the transitive closure $\mathcal{R}^*$. (It is a well-known relation.) Explain your answer.

5. Let $\sim$ be the relation on $\mathbb{R}$, the set of real numbers, such that for x and y in $\mathbb{R}$, $x \sim y$ if and only if $x - y \in \mathbb{Z}$. For example, $\sqrt{2} - 1 \sim \sqrt{2} + 17$ because the difference, $(\sqrt{2} - 1) - (\sqrt{2} + 17)$, is -18, which is an integer. Show that $\sim$ is an equivalence relation. Show that each equivalence class $[x]_\sim$ contains exactly one number a which satisfies $0 \le a < 1$. (Thus, the set of equivalence classes under $\sim$ is in one-to-one correspondence with the half-open interval $[0, 1)$.)

6. Let A and B be any sets, and suppose $f \colon A \to B$. Define a relation $\sim$ on B such that for any x and y in A, $x \sim y$ if and only if $f(x) = f(y)$. Show that $\sim$ is an equivalence relation on A.

7. Let $\mathbb{Z}^+$ be the set of positive integers $\{1, 2, 3, \ldots\}$. Define a binary relation $\mathcal{D}$ on $\mathbb{Z}^+$ such that for n and m in $\mathbb{Z}^+$, $n \,\mathcal{D}\, m$ if n divides evenly into m, with no remainder. Equivalently, $n \,\mathcal{D}\, m$ if n is a factor of m, that is, if there is a k in $\mathbb{Z}^+$ such that $m = nk$. Show that $\mathcal{D}$ is a partial order.

8. Consider the set $\mathbb{N} \times \mathbb{N}$, which consists of all ordered pairs of natural numbers. Since $\mathbb{N} \times \mathbb{N}$ is a set, it is possible to have binary relations on $\mathbb{N} \times \mathbb{N}$. Such a relation would be a subset of $(\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$. Define a binary relation $\preceq$ on $\mathbb{N} \times \mathbb{N}$ such that for (m, n) and (k, ℓ) in $\mathbb{N} \times \mathbb{N}$, $(m, n) \preceq (k, \ell)$ if and only if either $m < k$ or $((m = k) \land (n \le \ell))$. Which of the following are true?

a) $(2, 7) \preceq (5, 1)$ **b)** $(8, 5) \preceq (8, 0)$

c) $(0, 1) \preceq (0, 2)$ **d)** $(17, 17) \preceq (17, 17)$

Show that $\preceq$ is a total order on $\mathbb{N} \times \mathbb{N}$.

9. Let $\sim$ be the relation defined on $\mathbb{N} \times \mathbb{N}$ such that $(n, m) \sim (k, \ell)$ if and only if $n + \ell = m + k$. Show that $\sim$ is an equivalence relation.