Let $T(x, y)$ stand for "x has taken y," where the domain of discourse for x consists of students and the domain of discourse for y consists of math courses (at your school). Translate each of the following propositions into an unambiguous English sentence.

(a) $\forall x \forall y T(x, y)$

(b) $\forall y \exists x T(x, y)$

(c) $\exists y \forall x T(x, y)$

Answer:

(a) Every student has taken every math course.

(b) Every math course has been taken by some student.

(c) There is a math course that every student has taken.

Discussion: It can be helpful to start with a fairly direct, if awkward, translation of the symbols and then rework that into a more natural-sounding sentence.

(a) For all students x, for all math courses y, x has taken y.
For all students x, x has taken all of the math courses.
Every student has taken every math course.

(b) For all math courses y, there exists a student that has taken y.
Every math course has been taken by some student.

(c) There exists a math course y where for all students x, x has taken y.
There is a math course that every student has taken.

Translate each of the following sentences into a proposition using predicate logic. Make up any predicates you need. State what each predicate means and what its domain of discourse is.

(a) All crows are black.

(b) Any white bird is not a crow.

Answer:

(a) Let $CROW(a)$ be "a is a crow" and $BLACK(a)$ be "a is black". The domain of discourse for both predicates is things.

$$\forall x (CROW(x) \rightarrow BLACK(x))$$

For all things which are crows, they are black.

(b) Let $BIRD(a)$ be "a is a bird", $CROW(a)$ be "a is a crow", and $WHITE(a)$ be "a is white". The domain of discourse for all predicates is things.

$$\forall x(WHITE(x) \wedge BIRD(x) \rightarrow \neg CROW(x))$$

For all things which are white birds, they are not crows.

Discussion:

(a) The domain of discourse for both predicates is what is commonly understood for those predicates — essentially things and things which have a color. (In this case, it's not really necessary to explicitly state the domain of discourse.)

Considered for all things, this proposition will be true if those things that are crows are also black and false if a thing that is a crow isn't black, which is consistent with the original statement. The blackness or lack thereof of non-crows has no bearing on the truth of the original statement, so the proposition being true for non-crows is also consistent with that.

An alternate translation is $\forall x BLACK(x)$ where $BLACK(a)$ is the predicate "a is black" and the domain of discourse is "crows". But this is less satisfactory because it limits the application of "is black" in a way that goes against the typical concept — lots of things can be black, not just crows.

(b) While "any white bird" may sound like you just need to pick one white bird, it is actually conveying the idea that "not a crow" applies no matter which white bird you pick. Thus "not a crow" most apply to all white birds or else there's a chance that the particular white bird you picked is actually a crow.

Simplify each of the following propositions. In your answer, the $\neg$ operator should be applied only to individual predicates.

(a) $\neg\forall x(\neg P(x))$

(b) $\neg\forall z(P(z) \rightarrow Q(z))$

Answer:

(a)
$$\begin{aligned} \neg\forall x(\neg P(x)) &\equiv \exists x\neg(\neg P(x)) && \text{DeMorgan's Law for predicate logic} \\ &\equiv \exists x P(x) && \text{Double Negation Law} \end{aligned}$$

(b)
$$\begin{aligned} \neg\forall z(P(z) \rightarrow Q(z)) &\equiv \neg\forall z(\neg P(z) \vee Q(z)) && \text{definition of } \rightarrow \\ &\equiv \exists z\neg(\neg P(z) \vee Q(z)) && \text{DeMorgan's Law for predicate logic} \\ &\equiv \exists z(P(z) \wedge \neg Q(z))) && \text{DeMorgan's Law} \end{aligned}$$