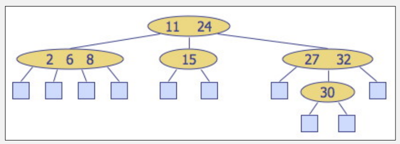
Multiway Search Trees

A *multiway search tree* allows more than one value per node.

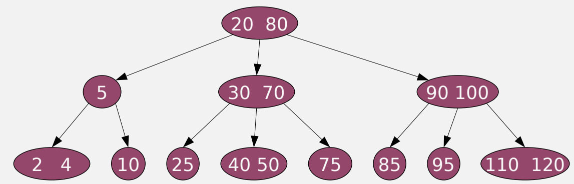
- generalization of binary tree
  - *order m* allows each node to have up to *m*-1 values and *m* children and *m*-1 values has up to m-1 values, in sorted order
- a node with *k* values has *k*+1 children (which may be empty)
- *i*th subtree of a node $[v_1, \ldots, v_k]$ only contains values in the range $v_i \leq v < v_{i+1}$
  - $0 \leq i \leq k$
  - $v_o = -\infty$, $v_{k+1} = \infty$



2-4 Trees

A *2-4 tree* is a multiway search tree where

- all leaves are at the same depth
- each node has 1, 2, or 3 keys and (# keys)+1 children



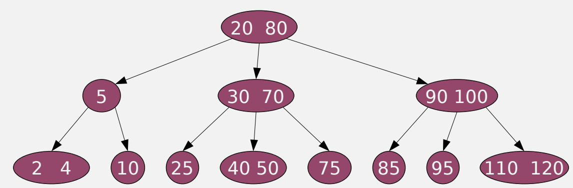
Height of 2-4 Trees

Does this ensure logarithmic height?
→ Yes!

Observe.

- the 2-4 tree with the fewest keys for its height has 1 key per node (complete binary tree)
  - level i has 2^i keys and the whole tree has $n = 2^h-1$ keys
    → h = O(log n)
- the 2-4 tree with the most keys for its height has 3 keys per node
  - level i has 3×4^i keys and the whole tree has $n = 4^h-1$ keys
    → h = O(log n)

Operations on 2-4 Trees



Searching in a multiway tree is similar to searching in a binary tree –
if the target element is not one of the keys in the current node, continue the search with the appropriate child.

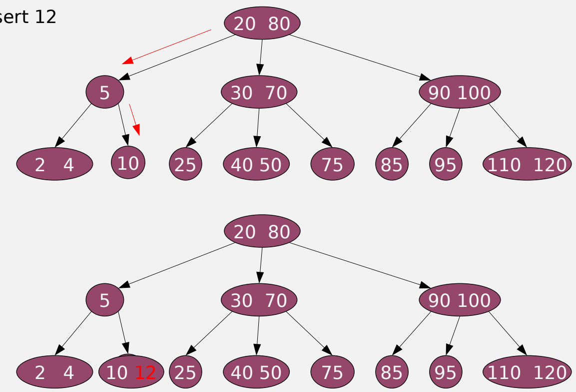
Operations on 2-4 Trees

For insert and remove, we use the same approach as with AVL trees:

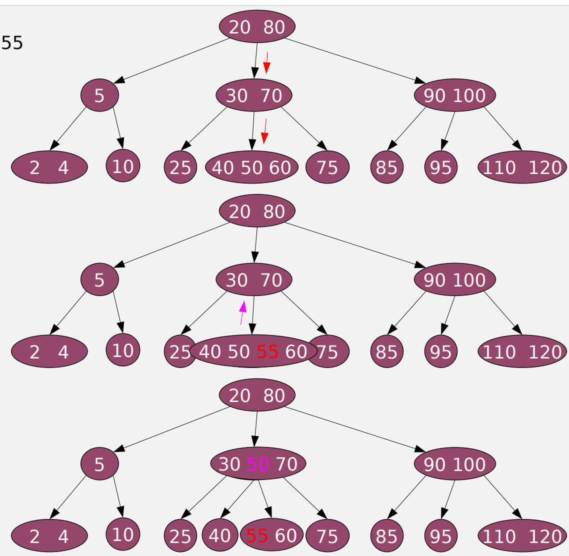
- insert/remove as dictated by the structural and ordering rules
  - new elements are always inserted at a leaf
  - elements can only be removed from a leaf – first swap with next larger (or smaller) as needed
- fix up the broken node size property as needed
  - if insertion creates an overflow (too many keys) –
    - split the node and promote a middle item to the proper place in the parent
    - repeat until there are no more overflows, creating a new root if necessary
  - if removal creates an underflow (not enough keys) –
    - if there's a sibling with at least two keys, transfer one (via the parent)
    - otherwise, merge – move a key from the parent, merging the node with a sibling
    - repeat until there are no more underflows, removing the root if necessary

Insert

insert 12

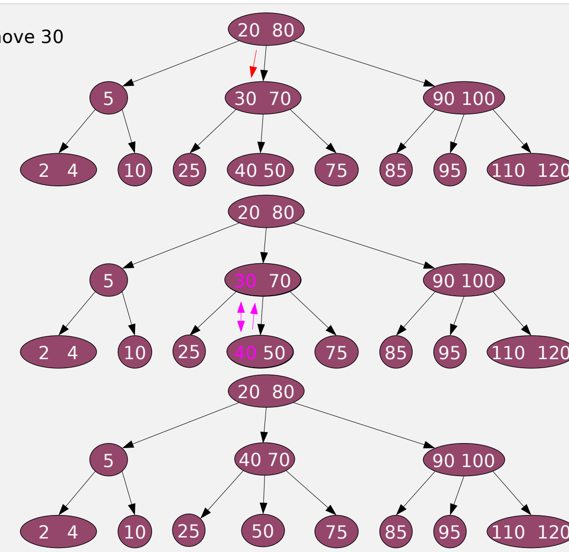


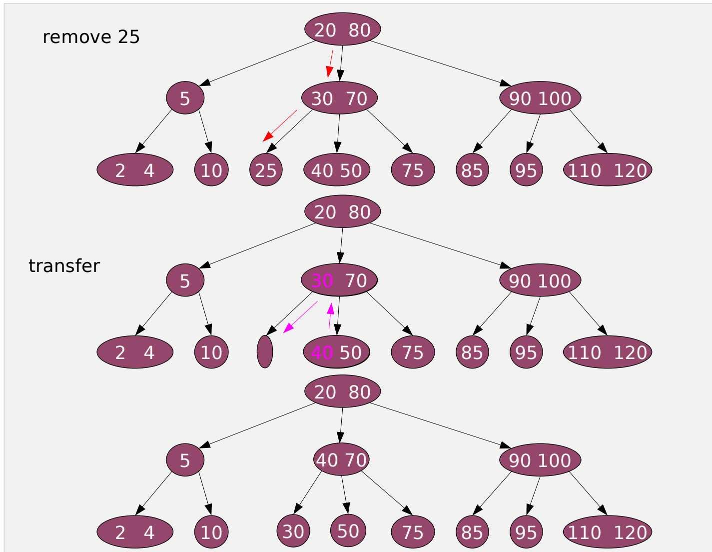
insert 55

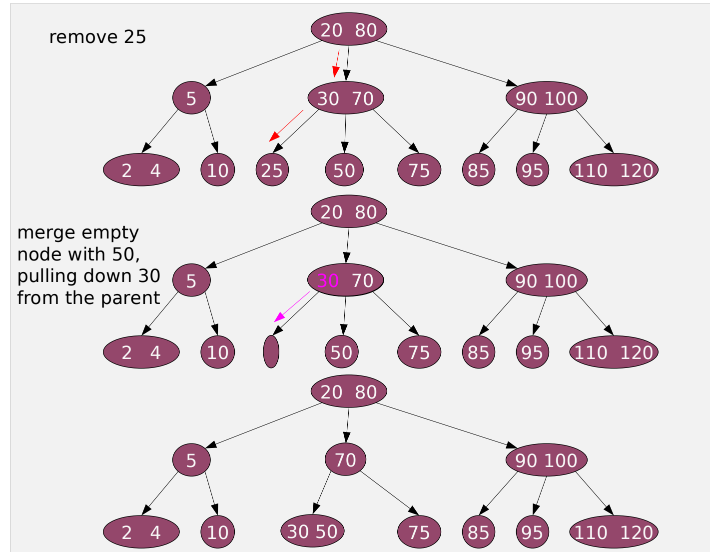


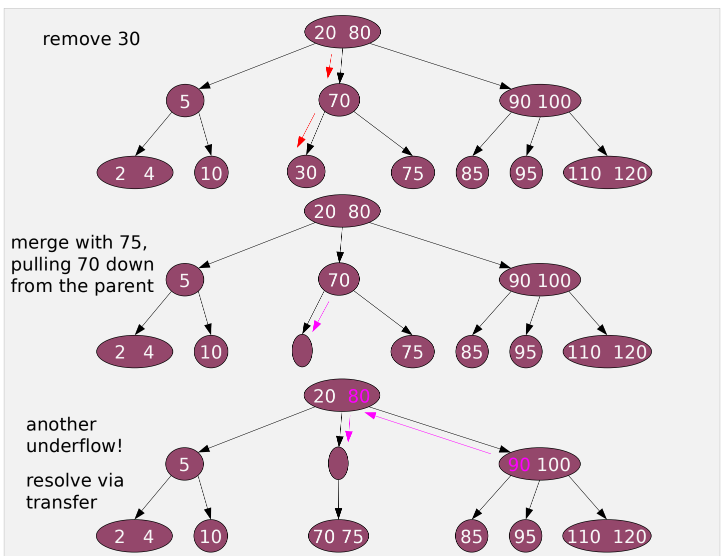
remove 30

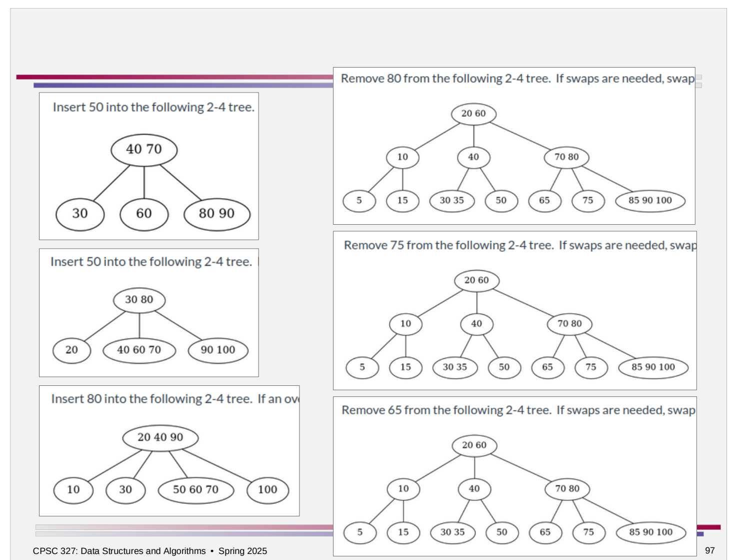
swap




remove 25
transfer


remove 25
merge empty node with 50, pulling down 30 from the parent


remove 30
merge with 75, pulling 70 down from the parent
another underflow!
resolve via transfer


Insert 50 into the following 2-4 tree.
Insert 50 into the following 2-4 tree.
Insert 80 into the following 2-4 tree. If an ov
Remove 80 from the following 2-4 tree. If swaps are needed, swap
Remove 75 from the following 2-4 tree. If swaps are needed, swap
Remove 65 from the following 2-4 tree. If swaps are needed, swap
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2-4 Trees Running Time

- time for initial insert – O(log n)
- time to fix up one overflow – O(1)
- number of overflows to fix – O(log n)

→ total time for insert – O(log n)

- time for initial remove – O(log n)
- time to fix up one underflow – O(1)
- number of underflows to fix – O(log n)

→ total time for remove – O(log n)

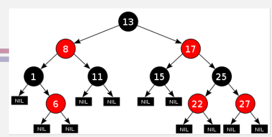
Balanced Search Trees

2-4 trees achieve O(log n) height by fixing the maximum depth of any element. This is made possible by allowing flexibility in the number of elements per node.

Red-black trees have the same idea – logarithmic max depth – but achieve it through flexibility in height rather than in the number of elements per node.

- structurally equivalent to 2-4 trees
  - can create an instance of the other with elements in the same order
  - can map operations on one into operations on the other

Red-Black Trees



A red-black tree is a BST + coloring rules:

- the root and the (null) leaves are black
- every red node has two black child nodes
- every path from a node to any of its descendant leaves contains the same number of black nodes

Properties.

- O(log n) height
  - longest root-to-leaf path (alternating red and black nodes) is no more than twice as long as the shortest (all black nodes)
- O(log n) insert/remove
  - O(log n) to perform insert/remove
  - O(log n) color changes and at most three restructurings to restore properties

Splay Trees


- invented by Daniel Sleator and Robert Tarjan in 1985

A *splay tree* is a BST + a restructuring operation:

- after each find/insert/remove, that node (or its parent) is brought to the root through *splaying*

Observation.

- frequently-accessed nodes are near the root

Does this ensure O(log n) height?

- on average, yes
- worst case is O(n) – but the worst case is unlikely

Splaying

- *x* is the node being splayed
- *p* is the parent of *x*
- *g* is the parent of *p* (i.e. the grandparent of *x*)

Case 1: zig – applies when *p* is the root

- x is rotated to the root

http://en.wikipedia.org/wiki/Splay_tree

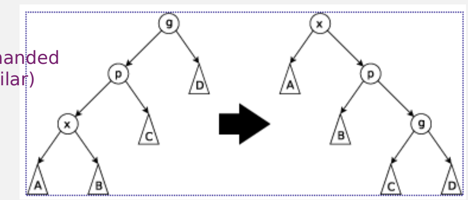
Splaying

- *x* is the node being splayed
- *p* is the parent of *x*
- *g* is the parent of *p* (i.e. the grandparent of *x*)

Case 2: zig-zig – applies when *p* is not the root, and *x* and *p* are both either right children or left children

- p is rotated into g's position, then x is rotated into p's position

(the right-handed case is similar)



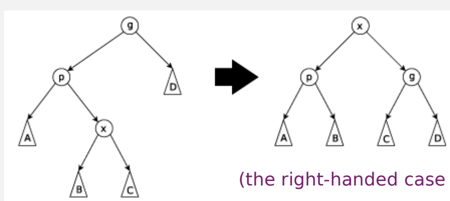
http://en.wikipedia.org/wiki/Splay_tree

Splaying

- *x* is the node being splayed
- *p* is the parent of *x*
- *g* is the parent of *p* (i.e. the grandparent of *x*)

Case 3: zig-zag – applies when *p* is not the root, and one of *x* and *p* is a right child and the other is a left child

- x is rotated into p's position, then x is rotated into g's position



(the right-handed case is similar)

http://en.wikipedia.org/wiki/Splay_tree

Performance

- all operations are O(height) to perform the operation + O(height) splay steps
  - each zig-zig or zig-zag raises *x* two levels, each zig (done at most one per splay) raises *x* one level

  → O(log n) amortized

- worst-case performance
  - splay trees perform as well as optimum static balanced BSTs on sequences of at least *n* accesses (up to a constant factor)
    - “static” = no restructuring of tree after construction
    - “optimal” = tree providing smallest possible time for a series of accesses
  - it is conjectured that splay trees perform as well as optimum dynamic balanced BSTs on sequences of at least *n* accesses (up to a constant factor)
    - “dynamic” = tree can be restructured after construction (e.g. AVL trees, red-black trees)

Splay Trees Takeaways

- another form of restructuring operation
- randomized or heuristic approaches can result in good performance in practice because worst case scenarios are rare
- amortized analysis
  - based on performance over a series of operations

Comparison

AVL trees and splay trees achieve O(log n) height by keeping the subtrees from getting too uneven.

2-4 trees and red-black trees achieve O(log n) height by constraining the maximum depth of any element.

- AVL trees, 2-4 trees, and red-black trees all have worst-case O(log n) operations
- splay trees have amortized O(log n) operations, with worst-case O(n) behavior

Comparison

- frequently-accessed elements are near the root in splay trees, resulting in faster access
  - advantageous in applications where there is *locality of reference* – repeated access of related storage locations
- AVL trees are more tightly balanced than red-black trees, so faster retrieval but slower insertion and removal
  - "tightly balanced" → smaller height
  - AVL trees are good for applications where trees are built once but searched often

Comparison

- splay trees have the simplest implementation
  - just BST + restructuring operation – no additional information to store/maintain
- red-black trees are more commonly used than 2-4 trees
  - easily built on top of binary trees – just need to store a color bit
  - simpler to implement than 2-4 trees
- 'find' may restructure a splay tree
  - from a design perspective, having read-only operations change structure is undesirable
  - a consequence is that splay trees are not thread-safe for concurrent finds without extra bookkeeping

Designing Data Structures

Studying balanced search trees reveals two tactics:

- it can be effective to add additional properties to the organization of the elements stored in order to improve runtime of an operation
  - e.g. AVL trees, 2-4 trees, red-black trees
- it can be effective (though harder to analyze) to piggyback local optimizations on other operations
  - e.g. splay trees

In both cases, it is essential that the additional work does not overwhelm the savings gained.