Graph Traversal

Building blocks and observations –

- Graph ADT provides operations for getting edges incident on a vertex, and end vertices of an edge
  - from a vertex you can find edges, and from an edge you can find the vertex at the other end
- there may be more than one vertex adjacent to another, so you can't just trace through the graph using a single finger to point at where you are – need a container to hold *discovered* vertices

Using a ~~queue~~ stack for the container leads to ~~breadth-first~~ depth-first search.

- however, DFS is typically implemented recursively rather than using a separate stack

Depth-First Search

```
dfs(G,s)      G is the graph, s is the starting vertex
  for each vertex u in V-{s} do
    state[u] = "undiscovered"
    prev[u] = null
  state[s] = "discovered"
  prev[s] = null
  dfshelper(G,s)

dfshelper(G,u)
  process vertex u (early)
  for each edge (u,v) in G.incidentEdges(u) do
    if state[v] = "undiscovered" then
      process edge (u,v)
      state[v] = "discovered"
      prev[v] = u
      dfshelper(G,v)
  state[u] = "processed"
  process vertex u (late)
```

this is a generalized form of the algorithm which allows for both early (before visiting incident edges) and late (after visiting incident edges) operations

a vertex is discovered when an incident (incoming) edge has been visited – have found a path from *s* to it

a vertex is processed when all of its incident (outgoing) edges have been visited – have found everything reachable from it

DFS

```
dfs(G,s)
  for each vertex u in V-{s} do
    state[u] = "undiscovered"
    prev[u] = null
  state[s] = "discovered"
  prev[s] = null
  dfshelper(G,s)

dfshelper(G,u)
  process vertex u (early)
  for each edge (u,v) in G.incidentEdges(u) do
    if state[v] = "undiscovered" then
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      prev[v] = u
      dfshelper(G,v)
  state[u] = "processed"
  process vertex u (late)
```

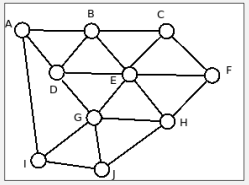

incidentEdges(u) determines what order the edges are visited in

the recursion keeps track of where the algorithm is in the sequence – execution continues when the call returns

Running Time of DFS

total O(n+m) for adjacency list, $O(n^2)$ for adjacency matrix

```
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  for each vertex u in V-{s} do
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  process vertex u (late)
```

O(n) with O(n) traversal of vertices and O(1) labeling

incident edges is O(deg(u)) for adjacency list, O(n) for adjacency matrix

total is O(m) for adjacency list (each edge is visited twice, once from each end) and $O(n^2)$ for adjacency matrix (get incident edges once per vertex)

Applications of DFS – Undirected Graphs

- **reachability**
  - every vertex reachable from *s* will be discovered/processed during DFS

intuition – we follow every edge leaving each discovered vertex, and every vertex put in the stack is eventually removed and marked as processed

```
dfs(G,s)
  for each vertex u in V-{s} do
    state[u] = "undiscovered"
    prev[u] = null
  state[s] = "discovered"
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  dfshelper(G,s)

dfshelper(G,u)
  process vertex u (early)
  for each edge (u,v) in G.incidentEdges(u) do
    if state[v] = "undiscovered" then
      process edge (u,v)
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      prev[v] = u
      dfshelper(G,v)
  state[u] = "processed"
  process vertex u (late)
```


BFS/DFS Search Trees

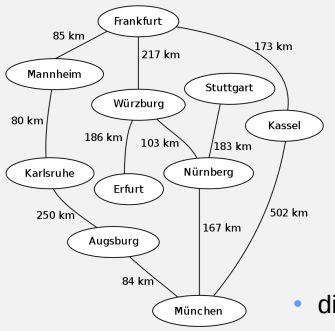
Classify each graph edge (*u*,*v*) as it is visited during traversal –

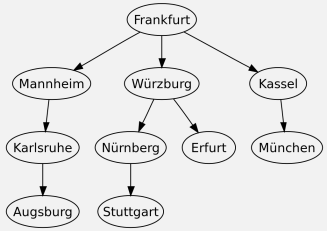
- *discovery* or *tree edges* – *v* is not already discovered
  - `prev` labels identify these edges
- *back edges* – *v* is an ancestor (other than the parent) of *u*
- *forward edges* – *v* is a descendant of *u*
- *cross edges* – *v* is not an ancestor or a descendant of *u*

Properties (undirected graphs) –

- discovery (tree) edges form a tree
- non-tree edges in BFS tree are cross edges connecting to the same level or one level higher in another branch
- non-tree edges in DFS tree are back edges

BFS/DFS Search Trees



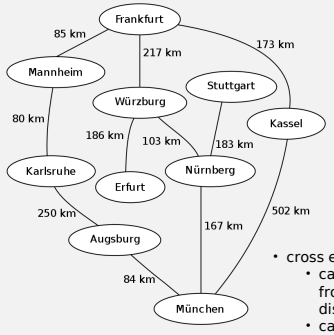


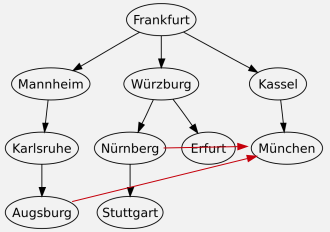
- discovery edges form a tree
  - a newly-discovered vertex is not already part of the structure so it can't be involved in a cycle

 https://en.wikipedia.org/wiki/Breadth-first_search

BFS/DFS Search Trees

- non-tree edges in BFS tree are cross edges connecting to the same level or one lower in another branch

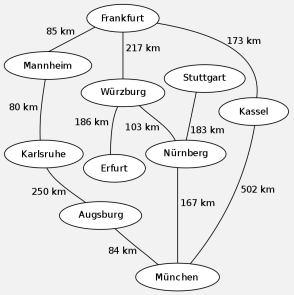


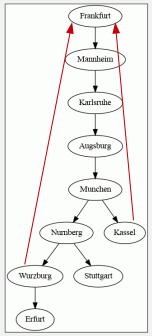


- cross edges can only connect to a vertex in the queue
  - can't connect to a vertex which has been removed from the queue or else it would have been a discovery edge from the other vertex
  - can't connect to a vertex not yet put in the queue or else would be a discovery edge from this vertex
- vertices in the queue at the same time can only be from adjacent levels

BFS/DFS Search Trees

- non-tree edges in DFS tree are back edges



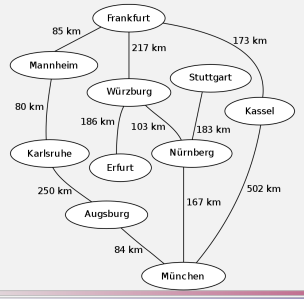


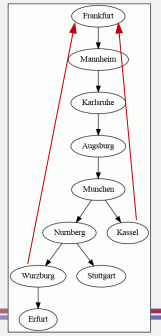
if (u,v) is a non-tree edge, v must be discovered but not processed...
  – if v has been processed, all of its incident edges have been visited and thus $(v,u) = (u,v)$ is a discovery edge

...meaning that v is an ancestor of $u \rightarrow (u,v)$ is a back edge
  – if v is the parent of u, $(v,u) = (u,v)$ is a discovery edge

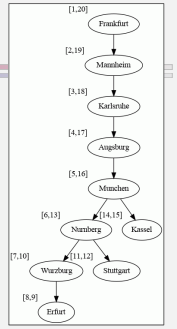
Applications of DFS – Undirected Graphs

- **finding cycles**
  - back edge (u,v) forms a cycle consisting of the tree edges from v to u plus back edge (u,v)
  - a graph is a tree if and only if there are no back edges





Entry and Exit Times



Recording entry and exit times –

- early process (before incident edges)

```
time = time+1
entry[v] = time
```

- late process (after incident edges)

```
time = time+1
exit[v] = time
```

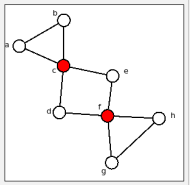
Properties –

- the [entry,exit] interval for v is properly nested within interval for ancestor u
  - entry times for ancestors of v are smaller than for v, while exit times are larger
- the number of descendants of v is (exit[v]-entry[v])/2
  - the [entry,exit] interval for all of the descendants is properly nested within the interval for v – so there is both an entry and an exit for each
  - time is incremented once for each entry and once for each exit

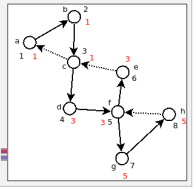
Applications of DFS – Undirected Graphs

- **articulation (cut) vertices**
  - a *cut vertex* is a vertex whose removal disconnects the graph (single point of failure)
  - a *biconnected graph* has no cut vertices (at least two vertices must be removed to disconnect)
  - observation – if a back edge connects a descendant of v with an ancestor of v, v is not a cut vertex
    - because the back edge forms a cycle
  - idea – for each vertex, determine its *earliest reachable ancestor* in the DFS search tree
    - number vertices in the order first encountered by DFS (entry time)
    - earliest reachable ancestor = lowest-numbered of v, the vertices adjacent to v via back edges, and the earliest reachable ancestors of children of v
    - v is a cut vertex if
      - the earliest reachable ancestor of at least one of v's children is the child itself or v
      - if v is the root, it must also have two or more children

cut vertices marked in red

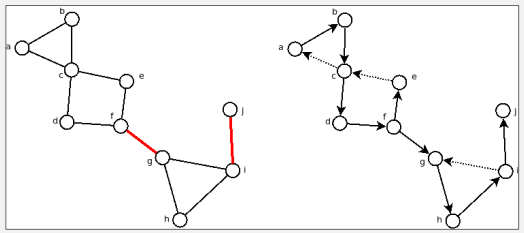


DFS tree – DFS entry order in black, earliest reachable ancestor in red



Applications of DFS – Undirected Graphs

- **bridges (cut edges)** – edges whose removal disconnects the graph
  - edge (*u*,*v*) is a cut edge if it is a tree edge and there's no back edge from *v* or a descendant of *v* to *u* or an ancestor of *u*



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DFS tree